THE SIXTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The sixteenth regular meeting of the Southwestern Section of this Society was held at the University of Missouri on Saturday, December 1, 1923. The total attendance was thirty-one, including the following eighteen members:

E. F. Allen, Ashton, Blumberg, Brenke, Chittenden, Decherd, Dunkel, E. R. Hedrick, Ingold, Lefschetz, J. V. McKelvey, U. G. Mitchell, Rider, Eugene Stephens, Stouffer, J. S. Turner, R. A. Wells, Westfall.

Professor Hedrick occupied the chair, being relieved during the sessions by Professors Ashton and Brenke. The morning session was devoted to the reading of the papers listed below.

During the afternoon session Professor Henry Blumberg gave a special lecture by invitation of the program committee on *Properties of unrestricted functions.* This lecture will appear in full in an early issue of this BULLETIN.

It was voted to hold the next meeting of the Southwestern Section at Iowa State College, at Ames, Iowa. The following program committee was elected: Professors E. R. Smith (chairman), R. L. Moore, and E. B. Stouffer (secretary).

The titles and abstracts of the papers read are given below. Professor Dickson's paper was read by Professor Hedrick. The papers of Mr. Michal and Professor Meacham were read by title.

1. Professor L. E. Dickson: *Integral solutions of the equation* $x^2 - my^2 = zw$.

This paper appeared in the December number of this BULLETIN.

2. Professors E. R. Hedrick and Louis Ingold: *Extensions of Beltrami's equations.*

The equations of Beltrami, which are sometimes used to define a complex function on a two-dimensional surface,

occur in connection with other two-dimensional problems. In this paper these problems and the corresponding Beltrami equations are extended to higher dimensions, in a manner consistent with the paper by the same authors presented to the Society at the Summer Meeting.

3. Professor Louis Ingold: *A symbolic treatment of the geometry of hyperspace.*

In previous papers the author has given a vector interpretation of Maschke's symbols and symbolic differential parameters, with applications to the properties of tangent varieties to arbitrary spaces characterized by a quadratic differential form. In this paper the symbolic method with the vector interpretation is used to study the normal and curvature properties of such spaces.

In addition to the geometric results, a number of new relations are obtained connecting various differential parameters and invariants.

4. Mr. A. D. Michal: *Integro-differential invariants of one-parameter groups of Fredholm transformations.*

The author finds necessary and sufficient conditions that an analytic functional of a function y and its derivative be invariant under a given arbitrary continuous one-parameter group of linear functional transformations of Fredholm type of the argument y. Further, it is shown that if the kernel of the infinitesimal transformation is of a certain type, such invariant functionals always exist. The calculation of the invariants in this latter case is then effected.

The functional is assumed to involve y and its derivative between the values 0 and 1, and, considered as a functional of these two arguments, may, for a given functional value of the derivative of y, be assigned arbitrarily as a functional of the other argument.

5. Professor S. Lefschetz: *The Kronecker-Poincaré index for certain manifolds.*

In this paper it is shown that the Kronecker-Poincaré index of two manifolds representing in function S_4 the zeros of two functions of two complex variables is equal to their number of intersections. The theorem and its generalization give rise to important and varied applications in algebraic geometry.

6. Professor J. S. Turner: *Rational triangles in which one angle is a rational multiple of another.*

In this paper, a simple method is described for finding the relation between the sides of a triangle ABC when $A = nB$, where n is rational. The relation is computed, and the complete integral solution (which may also be interpreted as the complete rational solution) is given, when $n = 2, 3, 4, 5$ and $^5/_2$.

7. Dr. E. F. Allen: *The jacobian of a contact transformation.*

If $x_1 = X(x,y,p)$, $y_1 = Y(x,y,p)$, $p_1 = P(x,y,p)$ is a contact transformation, it satisfies the Pfaff differential equation $dy_1 - p_1 dx_1 = \varrho(dy - p\,dx)$, where ϱ is in general a function of x, y and p. It is shown that the jacobian of this transformation is equal to ϱ^2. This is generalized to a space of $n+1$ dimensions. It is also shown that the surface $\varrho = 0$ in x, y, p space is transformed into a curve in x_1, y_1, p_1 space. This is also generalized to a space of $n+1$ dimensions.

8. Professor E. B. Stouffer: *On polynomials expressed as determinants with linear elements.*

Dickson has determined the types of general homogeneous polynomials which can be expressed as determinants with linear elements. In the present paper the author obtains Dickson's main results by entirely different means, use being made of the independence of sums of principal minors of determinants.

The method here used gives a simple proof of the well known theorem: *A sufficiently general cubic surface can be generated by three projective bundles of planes.*

9. Professor E. D. Meacham: *Surfaces whose osculating ruled surfaces belong to linear complexes.*

The necessary and sufficient conditions that a ruled surface belong to a linear complex have been given by Wilczynski. Making use of these conditions, the author of this paper investigates those non-ruled surfaces whose osculating ruled surfaces are contained in such linear complexes.

Certain properties of these surfaces and of the linear complexes involved are established, and a geometric construction of the surfaces is given.

10. Professor E. D. Meacham: *Notes on the geometric characterization of the two lines common to four linear complexes associated with a point of a non-ruled surface.*

The author proves the following theorem: At a point P on a non-ruled surface S, the two lines common to the four fundamental complexes are the diagonals of a skew quadrilateral whose sides are the four flecnode tangents and whose vertices are the focal points on these sides.

11. Professor E. W. Chittenden: *Nuclear and hypernuclear points in the theory of abstract sets.*

A point p is a nuclear point of a set E of power μ if every neighborhood of p contains a subset of E of power μ; hypernuclear if there exists for every neighborhood V of p a subset of E which is *interior* to V. The paper states in terms of these concepts necessary and sufficient conditions that a compact set in a space V be perfectly compact, and that a self-compact set possess the property of Borel-Lebesgue. The second theorem generalizes theorems of Fréchet and Kuratowski-Sierpinski, and is of special interest because it is shown by an example to be independent of the closure of derived sets.

E. B. STOUFFER,
Secretary of the Section.

PROBLEMS IN INVOLUTORIAL TRANSFORMATIONS OF SPACE*

BY VIRGIL SNYDER

1. *Introduction.* A report of great value and of general interest was presented to this Society at the Chicago meeting of April, 1922. While only a limited number had the opportunity to hear Professor Coble on that occasion, fortunately his message has reached a much wider public, by appearing in this BULLETIN (vol. 28 (1922), pp. 329–364). On account of the wider purpose there in hand, it was impossible to treat in detail all the many ramifications of the theory, or to show all their interrelations. My present purpose is to comment more fully on one narrow phase of this report, namely, that of involutions.

In the plane the problem is almost completely solved. It was shown by Bertini(1)† that there are four types to one of which every plane involutorial transformation can be reduced; they are the harmonic homology (H), the Geiser(2) (G) of order 8 with 7 triple points, the Bertini(1) (B) of order 17 with 8 six-fold points, and the Jonquières(3) (J) of order n with one fundamental point of order $n-1$ and $2n-2$ simple ones. Of these, all but the last are individual types, but (J) exists for every positive integral value of n. For special (J), JH is also an involution, but this can always be reduced to an H. Moreover, all these involutions are rational, that is, the pairs of conjugate points can be mapped rationally upon a plane (x') such that between (x') and the given plane (x) there exists an algebraic (1, 2) point correspondence.

Thus, that associated with (H) may be expressed in the form $x_2'x_3 - x_3'x_2 = 0$, $x_1'x_2x_3 - x_2'x_1^2 = 0$. The invariant

*Presented to the Society at the Symposium held in New York City, December 28, 1923.

†The numbers refer to the papers listed at the end of this article.

or coincident points constitute the line $x_1 = 0$. That associated with (G) has the form

$$x_1' C_1 + x_2' C_2 + x_3' C_3 = 0,$$
$$x_1' x_1 + x_2' x_2 + x_3' x_3 = 0,$$

in which each $C_i = 0$ is a general conic. Each cubic of the net formed by linear combinations of $x_i C_k - x_k C_i = 0$ is transformed into itself. The curve of invariant points is the jacobian of the net, $K_6 (r_1^2 \cdots r_7^2)$, in which r_i is a simple basis point. The curve of branch points in (x'), in $(1, 1)$ correspondence with K_6, is the general quartic L_4', which expresses the condition that the line in (x) touches the associated conic. Every line of (x) contains one pair of conjugate points. From this standpoint the system of the bitangents, the contact conics, cubics, etc. of L_4' can all be determined directly by elementary methods. Their two images in (x) are conjugate as to (G). That associated with (B) may be expressed by the equations

$$x_1' \varphi_1 + x_2' \varphi_2 = 0,$$
$$x_1' f + x_3' \varphi_1 (k_1 \varphi_1 + k_2 \varphi_2) = 0,$$

in which $\varphi_i = 0$ is a general cubic curve, and $f = 0$ is a proper sextic having eight of the points h_i common to $\varphi_1 = 0$, $\varphi_2 = 0$ for double points. The curve of invariant points is the jacobian of the system $\varphi_1 = 0$, $\varphi_2 = 0$, $f = 0$. It is of order 9, and has eight h_i as triple points, $K_9 (h_1^3 \cdots h_8^3)$. The curve of branch points is also a sextic L_6', having three branches touching each other at a common point. An arbitrary line in (x) contains four pairs of conjugate points; i. e., the simplest form of (B) is of class 4. Finally, the $(1, 2)$ correspondence of form (J) can be expressed by $x_1' x_2 - x_2' x_1 = 0$, $x_3' M_1 - u' M_2 = 0$, in which $M_i = 0$ is a curve of order m with an $(m-2)$-fold point at $(0, 0, 1)$; u' is linear in (x'). The jacobian is the curve of coincidences, a hyperelliptic curve having an $(n-2)$-fold point at $(0, 0, 1)$. This involution is of class zero. This is the easiest way to find and to classify the plane involutions of order 2; to find all possible ways in which the curves of a net can have two variable points of intersection. The question of reducibility or of the equiva-

lence of two given cases is not more difficult by this method than by that of Bertini. The seven basis points r_i have for images in (x') seven bitangents of L_4'; a point of a bitangent has two images, one of which is the basis point r_i, and the other describes the cubic of the net having a node at r_i. The point and the nodal cubic are conjugate in the involution (G).

Indeed plane involutions of every order were later shown to be rational, by Castelnuovo(4). For particular positions of the basis points the types assume special forms, but they are all included in the same (or simpler) categories(5). An interesting case of (J) is that in which all the curves of the net have contact of the second order on each branch at the multiple point. Its equation is

$$\lambda_1(-u_{n-1}x_3+u_n)+\lambda_2u_{n-1}x_1+\lambda_3u_{n-1}x_2=0$$

in which u_i is binary in x_1, x_2 of order i. All the basis points are now coincident at (0, 0, 1).

The curves of coincident points characterize the type of the involution. Hence it follows that any surface mapable on a double plane is, or is not, rational according as its curve of branch points can be reduced to one of the above curves(6). Another possibility is that involutions may be compounded. This is particularly useful in studying irrational algebraic curves. A curve is hyperelliptic if it has a linear series g_2^1. Thus every hyperelliptic curve is invariant under at least one involution. But a curve may be cut by the curves of a pencil in such a way that each curve cuts more than one pair of conjugate points from the given one. Such involutions, possible only on curves of genus greater than one, are usually irrational; they exist for any given genus(7). No algebraic curve of genus greater than 1 can belong to an infinite group, either continuous(8) or discontinuous. An elliptic curve is invariant under an infinite series of operations which do not form a group. It has an infinite number of central involutions each with four fixed points, and three without fixed points. A rational curve is invariant under a three-parameter group, within which are ∞^2 involutions.

2. *Transformations in Space.* Of the linear transformations, only two are involutorial, the central and the axial involution. In the former we have a plane of invariant points, in the latter, two skew lines. The quadratic transformations may be defined as in the plane. They were all found in the early papers of Cayley(9), Cremona(10), and Noether(11). An elementary derivation is given by Snyder and Sisam(12). The involutorial forms are either perspective, such that every line of a bundle remains invariant, or the product of the perspective inversion and a central homology. A full discussion is given by Doehlemann(13). An application to the theory of electric images was given by Liouville(14). For the theory and the literature, see Doehlemann(13). One great application of the birational transformations of space is to the study of systems of curves lying on given surfaces. Thus, on the quadric, every algebraic curve can be expressed by means of an (a, b) correspondence, and only those plane curves which can be thus expressed are projections of space curves of the same order lying on a quadric. Since the surfaces of a homaloidal web are all rational, the transformation furnishes an immediate method of mapping the surface on a plane. On the other hand, this method does not furnish general criteria for determining whether a birational transformation is involutorial or not. Nearly all of the earlier involutions were found directly from particular geometric conditions imposed by the problem considered, and none have been obtained by following the procedure employed by Bertini for the plane involutions.

The surfaces of the system must satisfy two fundamental conditions; they must form a linear system with four independent surfaces; any three intersect in one variable point. Hence the surfaces must have points or curves common to all. But these criteria are common to all birational transformations.

3. *Rational Involutions.* Consider in space (x), a linear system of ∞^3 surfaces, a web, having the property that

any three not in a pencil intersect in two variable points. Let the web be $\sum a_i\varphi_i = 0$. By putting $x'_i = \varphi_i\,(i = 1,2,3,4)$ the surfaces can be mapped upon the double space (x'). This was first studied by De Paolis(15).

To the planes of (x') correspond the surfaces of the web $|\varphi|$, each of which remains invariant under the involution of associated points, images of the same point of (x'). To the lines of (x') correspond the curves (φ_i, φ_k), also invariant under I. These curves are either hyperelliptic or belong to sub-types of hyperelliptic curves, of genus 1 or 0. Each surface of the web $|\varphi|$ has then ∞^2 hyperelliptic curves, any two of which intersect in two variable points. The locus of points in (x') having the property that the two images in (x) coincide is the surface L' of branch points. The coincident points in (x) define a surface K, in (1, 1) correspondence with the points of L'. These surfaces may be replaced by curves or by points. The system of surfaces in (x'), images of the planes of (x), do not form a linear system, but have certain points and curves in common which are fundamental for the correspondence. A fundamental point P' may have one or both its images in (x) fundamental(16). If its images are a point P and a curve p, then P is a fundamental point of the associated involution I in (x). If both images are fundamental, the associated curve in (x) may be fundamental in different ways.

A complete enumeration of rational involutions would involve the various webs of surfaces with two variable intersections. While these are of course infinite, it is not known to what extent they can be reduced to a small number of families. Thus, in the plane, if we exclude the hyperelliptic curves with common multiple points there are only two nets of curves with two variable points, one of genus 1 and the other of genus 2. The equations concerning a Cremona net are then sufficient to complete the enumeration. When the point P' describes a locus (curve or surface), the images P_1, P_2 will each describe a locus. These two loci may coincide, or if distinct, they are con-

jugate in I. The necessary and sufficient condition that the two loci p_1, p_2 are distinct is that p' touch L' at every non-fundamental point(17). From this standpoint all the systems of bitangent planes, of contact quadrics, cubics, etc. of the general L' can be found immediately without the use of transcendental methods. In particular the system of contact surfaces of the Kummer surface can be derived from the web of quadrics through 6 points. The numerous formulas derived by De Paolis are not in the main of greatest usefulness on account of being expressed in too many unknowns. When the web is regular, and the basis elements independent, the Riemann-Roch theorem for surfaces,

$$r = p_a + n - \pi + 1,$$

applies, in which r, the dimension of the system of curves in which a fixed surface of the system is cut by the other surfaces of the web, is 2, n is the number of variable intersections of two curves of the system, and π is the genus of the variable curve. Hence this genus exceeds the arithmetic genus of the general surface of the web by unity. Since each curve is hyperelliptic, the number of coincidences is $2\pi + 2$, hence the order of L' is also $2\pi + 2$.

The general discussion of De Paolis was followed by Schoute(18), who confined his discussion to simple cases. Reye(19) gave a detailed synthetic study of the web defined by quadrics through 6 points, which has been treated by a number of writers since. Hudson(20) developed it from the dual of the previous standpoint; Eberhard(21) followed various systems of curves on the surface, to which I added several(22).

An involution is determined on a quartic surface with five basis points by the space cubics through these and a point P, but this may not be birational for all of space. In the case of the Weddle surface the points P, P' are collinear with the sixth node. Other cases are treated by Sturm(23), by Marletta(24), and by Baldus(16). The latter also discussed (1, 2) correspondences between irrational ruled surfaces, in

which fundamental elements appear which are not found in planar involutions. A particular (1, 2) correspondence associated with a net of cubics through a space quartic curve γ_4 and five coplanar basis points was studied by Pieri(25), primarily for the properties of L'. An interesting property appears here, as the variable curves are of genus 2. The system of curves is irregular since the basis points are not independent. Three cases cited by Pieri as belonging to distinct types are now known to be reducible to particular cases of the types considered by him. Another important feature is the appearance of an irrational fundamental curve. In the plane the image of a fundamental point is always a rational curve of order equal to the multiplicity of the point. But the net of surfaces of the web which pass through an arbitrary point of the plane π all pass through the cubic curve of the pencil determined by this point, the basis points A, and the four points of intersection of π and γ_4. The image of this cubic in (x') is a point, and the point describes a straight line as the cubic curve describes the pencil. The other image of P' is a point R in (x) which depends upon how P' is approached. To an arbitrary plane through P' corresponds a linear g_2^1 on the cubic curve. This property accentuates the difficulty of defining a type, as an irrational curve may be transformed into a point.

Another web of cubics was considered by Romano(26), that having three skew lines and four points for basis elements. This presents six fundamental lines of the second kind, such that in the associated involution the image of any point on one of them is the whole line passing through it.*

In every case, when simple basis points of the web appear, the image of each in (x') is a tangent plane to L'; the conjugate, in the associated involution, of the basis point is the

*In the later discussion of the same case by Sharpe and Snyder, TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 56, these lines are overlooked, so that the characteristics of the involution as there given are incomplete. Similar omissions occur in the later types defined by a web of cubic surfaces. Romano's paper was not known to the authors at that time.

surface of the web having a double point at that point. When the web has a simple basis curve, each point of the curve goes into a straight line in (x'), and the conjugate in I is a rational curve. These curves generate a surface, conjugate of the given curve, which is a multiple basis curve in the involution.

But when the section of the surfaces of a web by a fixed surface is completely accounted for by fundamental elements, various paradoxes may appear. This has not been completely worked out, especially when the section is made on an irrational non-ruled surface.

The problem of the existence of these webs depends upon formulas of postulation involving contact of higher order at multiple points. In the original memoir of Noether[27] the cases are excluded in which the tangent cones to the surfaces of the web at a multiple basis point are composite, and contacts of different sheets along branches of the curve is not considered. Notwithstanding the recent valuable additions made by Hudson[28], by Severi[29], and by Tummarello[30], little progress can be made in the problem of reducibility and the determination of the equivalence of two given involutions until more is known on this subject.

When all the points of intersection of a line or a curve with surfaces of the web are fixed, or are on fixed basis curves, the line is called a parasite by De Paolis. Since a net of surfaces pass through it, its image in (x') is a point. If the given curve belongs to a linear system, each curve of which is a parasite, the image point describes a straight line.

A satellite is a line or point of contact which insists on coming in, unbidden, when certain conditions are imposed. Consider the conical surface having a cuspidal edge $x_1^2 x_3 = x_2^3$, and the rational cone $x_1 x_3^{n-1} + A x_2^k x_3^{n-k} + \cdots = 0$ with undetermined coefficients. The number of lines of contact approaching coincidence with $x_1 = 0$, $x_2 = 0$, in the plane $x_1 = 0$ is always an odd number. If we impose the condition that one more line be counted, a second insists on accompanying it. This is true for a cone having any superlinear branch.

Satellites and parasitic branches will have to be considered

in view of fundamental elements of new and as yet little understood forms. The explanation given by De Paolis(15) is incorrect, and the relation between a line and its parasite may violate the theorem of order and multiplicity, as has recently been shown by Montesano(31) and by Tummarello(30).

4. *Involutions by Means of* (1, 2) *Correspondences.* The first comprehensive study of the (1, 2) correspondence from the standpoint of classification of associated involutions was made by Sharpe and me(32). After collecting the necessary formulas of postulation and equivalence we applied them to webs of surfaces of orders 3, 4, and 5. Two kinds of restrictions were made at the outset. Surfaces with higher point singularities, including monoids, were excluded, and the basis elements were assumed to be independent. The first restriction was deliberate. When the surfaces are monoids or, more generally, rational, then each surface of the web can be mapped birationally upon the plane, and the curves of intersection with the other surfaces can be treated by means of plane involutions. The two extreme cases of monoids, those having one non-composite basis curve, and those having a fixed tangent cone at the vertex have been determined since, but the results are not yet published. The former are not monoidal transformations, and most of them cannot be reduced to the monoidal type. Those of the other form are always monoidal, of a very particular type. The other restriction was less deliberate. In the plane the basis elements are independent, and the fundamental curves are uniquely fixed by the position and multiplicity of the basis points through which they are to pass. If the elements are chosen in a particular way, for example three collinear points among the seven which define a Geiser net, the resulting involution can be reduced to a simpler form in which the elements are independent.

Within these limitations, the webs of cubic, quartic and quintic surfaces have been determined. Among them appear the focal surfaces of all the line congruences of order 2 and

class 2 to 7, and several related surfaces, together making an unbroken sequence. Incidentally, configurations of curves on these surfaces can be determined much more directly by this method than by those by which they were first discovered. The category also includes surfaces invariant under infinite discontinuous birational groups, both those known previously, as the Kummer surface and the Fano surface, and new surfaces not previously studied.

What was not accomplished was to determine how many of the involutions appeared more than once in a transformed form. It was proved, for example, that there are different involutions having a Kummer surface for surface of branch points L' in (x'). Each focal surface of a line congruence of order 2 appears at least twice. Of course one will meet, in the study of webs of higher order, all the involutions associated with simpler webs, which can be transformed birationally into webs of the given order. Thus, Osborn(33) has studied the webs of order 6, and has refound many of the existing types. The webs of singular cubics have recently been studied by C. Moffa(34) by the method of plane mapping without the use of the double space, and the results in the other webs confirmed by the same method. Miss Moffa shows that fundamental lines of the second kind must exist in every case. This paper, together with the correction to the result obtained by Pieri(25), completes the list of involutions which leave each surface of a web of cubics invariant.

5. *Bases of Classification.* Another scheme of classification is to divide the involutions into families, expressed as a function of the order n, like the Jonquières forms in the plane. The first attempt of this kind was made by De Paolis(35), who classified all the forms that leave a line congruence of order one invariant. The lines may belong to a bundle, or meet a line and a rational curve of order m, with $m-1$ points on the line, or be the bisecants of a cubic curve. The second and third cases are treated exhaustively, but from the standpoint of classification the results are of little

importance, as all of them can be reduced to the monoidal form. The involution of order 7 defined by a web of quadrics through six basis points belongs in this category. Most of its properties are much more easily obtained, however, in the form of the web of quadrics, than expressed in terms of the nodal projection of the Kummer surface upon itself.

Another important family was mentioned by Noether(36) and more fully treated by Montesano(37). In this family the images of the planes of space are surfaces of order n with a common $(n-2)$-fold line and a basis curve of order $3n-4$. This will be discussed later. Another family, also discussed by Montesano(38), consists of surfaces of order $2n+1$, having as basis elements a space elliptic quartic to multiplicity n, and $2n$ of its bisecants for simple lines. This case was also previously cited by Noether(39). It includes a number of particular cases previously mentioned by others. Thus, the type discussed by Pieri(25) is this one when $n=5$. The case in which the conjugates of planes are surfaces of order n with a fixed $(n-1)$-fold line was also studied by Montesano(40).

A number of families of involutions, generalizations of the preceding, were found by Sharpe and me(17), the results of which can perhaps be best expressed by the theorem: There exists an involution of order $12n+5$ having a fundamental quartic γ_4 of genus 1 to multiplicity $6n-1$ and a fundamental curve β_{6n+1}, genus $9n-3$, to multiplicity 4, meeting γ_4 in $12n-4$ points. It has also $12n+16$ simple fundamental lines meeting γ_4, β_{6n+1} each twice and $6n-2$ fundamental double conics meeting γ_4, β_{6n+1} each in 4 points. In symbols,

$$s_1 \sim s_{12n+5} : \gamma_4^{6n-1} + \beta_{6n+1}^4 + (12n+16)u_1 + (6n-2)u_2^2,$$
$$[\gamma_4, \beta_{6n+1}] = 12n-4;\ [u_i, \gamma_4] = [u_i, \beta_{6n+1}] = 2.$$

Similarly,

$$s_1 \sim s_{4n+9} : \gamma_1^{4n+1} + \beta_{5n+5}^3 + (15n+10)u_1 + 2u_2^2,$$
$$s_1 \sim s_{6n+9} : \gamma_1^{2n+6} + \beta_8^{2n+1} + \beta_{7n}^2,$$
$$s_1 \sim s_{6n+17} : \gamma_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}^2,$$
$$s_1 \sim s_{6n+10} : \gamma_2^{2(n-1)} + \gamma_7^{2n+2} + \beta_{7n-3}^2.$$

These various types are not reducible to each other.

6. *Monoidal Transformations.* The monoidal transformations were discussed by De Paolis(41) and the product of the perspective monoidal and a harmonic homology was treated by Martinetti(42), who thought there could be no other non-perspective types. All the forms were derived synthetically by Montesano(43), who showed that every type could be reduced to one in which the bundle of lines through the vertex remains invariant, and that the lines of the bundle are either invariant (perspective type) or are interchanged in pairs according to one of the four ternary involutions H, G, B, J. Of these, the latter contains two parameters, the order n of J, and the order m of the perspective monoid. The equations of all the types were derived by me(44).

Let $O = (0, 0, 0, 1)$ be the vertex of the monoid. Since the bundle of lines O is transformed into itself, and is involutorial, it must be either the identity, or H, G, B, or J. In the first case, called the perspective, the conjugate of any point P is on OP and is harmonic with regard to P and the pair of points in which OP meets a fixed surface

$$u_{n-2}x_4^2 + 2u_{n-1}x_4 + u_n = 0,$$

u_i being ternary in x_1, x_2, x_3 of degree i.

The equations of the transformation are

$$rx_i' = x_i, \quad i = 1, 2, 3; \quad rx_4' = \frac{-(u_{n-1}x_4 + u_n)}{u_{n-2}x_4 + u_{n-1}}.$$

Every plane through O is invariant and contains a perspective J involution from O. If u_{n-2} is identically zero, the fixed surface is itself a monoid, and P' is the harmonic conjugate of P as to O and the residual point of intersection with OP. The basis curve consists of the lines common to the two cones $u_{n-1} = 0$, $u_n = 0$, which are fundamental of the second kind, that is, the image of any point on any one of them is the whole line passing through it. The image of a point on a k-fold basis line is the whole line counted k-fold.

Of the other types, any one is expressible in the form

$$rx_i' = \varphi_i, \quad i = 1, 2, 3; \quad x_4' = \frac{x_4 \varphi u + v}{w x_4 - ux},$$

in which the φ_i define an involutorial Cremona net of cones with common vertex at O, φ is a linear function of the φ_i and x is the same linear function of the x_i. The u, v, w are algebraic polynomials in functions k_i that are invariant under the ternary involution.

In particular, take BM. The φ_i are of order 17, and have eight basis lines each to order 6. Now suppose $n = 6$, and v, w are identically zero. The result is of order 18, the image of O is the plane $x = 0$ passing through O, and every line joining a pair of conjugate points meets the fundamental line m: $x = 0, x_4 = 0$. The conjugate of an arbitrary line is a rational curve of order 18, which passes simply through O. If the given line meets m, then the image of the point of intersection is a component of the conjugate; the proper image is a plane curve of order 17 in the plane of m and the given line. It is a plane B transformation. The line meets the invariant cone $R_9 = 0$ of B in 9 points, which also lie on the conjugate C_{17}. Hence the line meets C_{17} in four pairs of conjugates. In this particular monoidal transformation any line joining a pair of conjugate points belongs to a special linear complex and contains three other pairs of conjugate points. This probably includes irrational involutions.

In connection with the monoidal types of birational transformations we have an immediate example of the existence of (n, n^2) types, found synthetically by Tummarello(45) for every value of n, although earlier writers supposed they could not exist for $n > 3$(46).

The equations of the transformation are of the form

$$x_i' = \varphi_i, \ (i = 1, 2, 3), \quad x_4' = u_{n-1}x_4 + u_n,$$

in which $|\varphi|$ is a Cremona net of cones of order n with vertices at $(0, 0, 0, 1)$.

7. *Classification.* The first proposal was to classify involutions, like other birational transformations, according to the order of the surfaces conjugate to the planes of space. This was done for those of order 2 in the earliest

memoirs. It was soon found that those of order 3 could be expressed by means of three symmetric bilinear equations in the coordinates of two variable conjugate points(47). The types arising from three polarities have been established by Morris(48), but a number of interesting additional types appear when one or two of the polarities are replaced by linear complexes, either one or both of which may be general or special. These types offer no new difficulty, but no enumeration is complete without them.

But this method was early recognized to be impracticable, as no general criteria are known, to insure the involutorial character of the transformation when $n>3$. A real advance was that of classifying involutions according to the genus of a general plane section of the conjugate surface of a plane(49). The first category is that of genus zero. The conjugate surface must be either a quadric, a Steiner quartic, or a ruled surface of order n with an $(n-1)$-fold line. Those of the first category were first obtained by Aschieri(50) and by Martinetti(42), those of the second by Montesano(51), and those of the third by Montesano(40). This method has not been developed further, on account of lack of knowledge of forms of rational surfaces with plane sections of given genus. Those of genus 1 have recently been made by Nobile(80).

8. *Line Complex containing the Involution.* The next procedure is that of considering the system of lines obtained by joining conjugate points. This system may be either a congruence or a complex. In the former case each line of the congruence contains an infinite number of pairs of conjugates. These congruences are always of order 1(35), and the transformations reducible to monoidal types. When the lines form a complex, each line contains one or more pairs of conjugate points. The linear complex, each line containing one pair, was considered by Montesano(52).

This does not give an extensive list of involutions, but each involution is determined by a particular curve of order 10 and genus 11. Given a hyperelliptic curve C_6^3 of

order 6 and genus 3. Two quartic surfaces through it determine a C_{10}^{11} through which pass ∞^4 quartic surfaces. Of this system, the web passing through a point P also intersects in its conjugate P'. The lines PP' generate a linear complex. An arbitrary plane meets C_{10}^{11} in 10 points lying on a cubic curve. The curve may break up into a particular(53) C_7^5 and a cubic. The involution is of order 11, and may be particularized to orders $10 \cdots 3$. Types belonging to a special linear complex were later studied by Pieri(54), but not all such types were found. They can all be expressed by a single formula, given by Sharpe and me(17). All of the monoidal types JM are included in this category, as well as the simple M.

Involutions belonging to the tetrahedral complex were derived by Montesano(55), and later from another standpoint by Wimmer(56). This type occurs incidentally in the discussion of Sharpe and Snyder(17). The involution is of order 19. Each surface of a web of quartics having a basis C_{11}^{14} is left invariant by the involution. Other cases of involutions belonging to special quadratic complexes were studied by Pieri(57), and of involutions belonging to complexes of tangents by Pieri(58), and by Montesano(59). They can all be reduced to monoidal types. The question now arises whether an involution belongs to every quadratic complex, and the answer is No. (60) The study of the complex to which an involution belongs is of great service, but it does not furnish a satisfactory basis for classification. Montesano(61) showed that involutions in which each conic of a linear system is self conjugate belonged to the type studied by De Paolis(35), hence reducible to monoidal forms.

In all these cases, each line of the complex contains a single pair of conjugate points, but involutions exist in which each line of the complex contains two, three, or four pairs of conjugate forms(44). Montesano has proposed to me to classify involutions according to the minimum product of the order of the complex to which it belongs and the number of pairs of points on each line of the complex.

9. *Irrational Involutions.* It is known that irrational involutions in space exist. The first example was given by Enriques(62), and later simplified by Aprile(63). But both of these are of order higher than two, and it is not known whether irrational involutions of order 2 exist or not. But there are three that make it seem very probable. The first was discussed by Montesano(64). Consider the group of order 32, $x_i' = \pm x_i$ which leaves the equations $\sum x_i^2 = 0$, $\sum a_i x_i^2 = 0$ invariant. Think of the x_i as line-coordinates, and map the quadratic complex defined by them on the points of space. The central perspective transformations go into six perspective monoidal ones of order 3, all the invariant cubic surfaces passing through a common space quintic of genus 2. The product of two of the central perspectives corresponds to cubic involutions which leave every plane through the line joining the centers invariant, and every point of an elliptic space quartic remains fixed, the curve being the residual to the quintic, intersection of the invariant cubic surfaces of the monoidal components. The ten products by threes ($123 = 456$) are more complicated. This depends upon the representation of the lines upon the point space, and the configuration is included in the next case, also considered by Montesano(37). The following outline is based upon a study being made by Professor Sharpe and me, not yet completed. Consider the involution I in which the conjugate of a plane is a surface of order n, having a line p to multiplicity $n-2$. There is also a basis curve of order $3n-4$ meeting p in $3n-7$ points. Planes through p are transformed into planes through p, either each into itself, or the pencil is in involution, with two invariant planes. If p is defined by $x_1 = 0$, $x_2 = 0$ we may write

$$x_1' = Mx_1, \quad x_2' = -Mx_2, \quad x_3' = E, \quad x_4' = F.$$

Since the transformation is birational and involutorial, it follows that x_3, x_4 are connected with x_3', x_4' by a bilinear equation. It is now easy to get the equation of the complex

to which the involution belongs. It is always a particular form, being most nearly general for $n = 3$. It has been shown by Fano(65) that the general cubic complex is irrational. In some particular cases the spatial, surface, and arithmetic genera were all found to be zero, but these conditions to not insure the rationality of the involution, since a three-dimensional variety may have all its genera zero and still be irrational(65). The third illustration is furnished by the general cubic variety V in four space(67). Pass a plane through any point P on V and any line l lying on it. The plane section consists of l and a conic meeting it in two points, A, B. The lines AP, BP are tangent to V at points on l. Conversely, given any tangent to V at a point of l. It meets V in a residual point P, hence there is a $(1, 2)$ correspondence between the points of V, and its tangents at points of l. If the tangents are now mapped on the points of an auxiliary three space, we obtain an involution belonging to V(68). It is of order 6, and belongs to the type JM, hence also to a linear complex, but every line joining a pair of conjugate points contains one other pair. Thus it is mapped on a special linear complex doubly. But we have practically no criteria of rationality of varieties which can be mapped doubly on rational three spaces(66).

10. *Curves and Surfaces Invariant under Involutions.* In every rational involution every surface of a web remains invariant, and also the curves of intersection of pairs of surfaces of the web. In this way many properties of curves can be obtained which belong to given irrational involutions of any given genus. Associated with every curve of (x'), not a contact curve of L', is a curve of (x) which is invariant under the involution, the pairs of conjugate points belonging to the given curve in (x'). And we may have surfaces belonging to more than one web, each of which defines an involution. On this surface, the two involutions generate a group which may be finite or infinite. Numerous examples of such surfaces have been given(69), but many of them are

disposed of by the theorem of Enriques[70] that such a surface has a pencil of elliptic curves, except when $p_a = P_2 = 1$. In every case thus far found, all the operations can be expressed in terms of Cremona transformations. A surface having $p_a = P_2 = 1$ but not containing a pencil of elliptic curves was found by Fano[71], and its group discussed by Severi[72]. That it belongs to two webs connected with (1, 2) correspondence was shown by Sharpe[32] and me. A second example was found by us[73], having similar properties, defined by the web of quartic surfaces through a space curve of order 8 and genus 2. A detailed discussion of all these types, with particular reference to their groups, was given by Fano[74].

But another quartic surface exists[73], for which $p_a = P_2 = 1$, which has no elliptic curves, and is invariant under an infinite discontinuous group, but not generated by involutions[75]. The operations of this group may also be expressed by Cremona transformations. The surface is defined by containing a sextic curve of genus 3.

PROBLEMS.

1. What is the definition of a type of space involution?
2. Can rational involutions be classified in a finite number of types?
3. Do irrational involutions of order two exist?
4. Do all surfaces invariant under infinite discontinous groups belong to Cremona groups?

   For example, given a quartic surface with five nodes. The space cubic determined by them and a point P on the surface meets it again in P', defining an involution under which the surface is invariant. Can this be expressed as a Cremona transformation? Now take a quartic with six nodes. There are six such involutions. Do they generate an infinite group? These illustrations may be trivial, since the surface contains pencils of elliptic curves.
5. Can the equivalence of two given involutions be expressed in terms of rational invariants?

This is a short list, but will probably suffice for some time. Encouraged by recent contributions to the theory of postulation and equivalence, as given by Hilda Hudson(76) and by Severi, and illustrated by Tummarello and by Miss Moffa, and by the important theorem of Montesano(77) concerning multiple parasitic lines, let us continue to hope that a final answer may be found to some of them. The recent researches of Chisini(78) on the singularities of algebraic surfaces, and the proposal of Severi(79) concerning classification of space curves will be of fundamental assistance.

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A GENERALIZATION OF THE SYLLOGISM*

BY B. A. BERNSTEIN

The syllogism is the proposition:

If x is y and y is z, then x is z.

In the language of boolean algebra this proposition is:

If $xy' = 0$ and $yz' = 0$, then $xz' = 0$,

where the usual notations are used, with the prime indicating negation. There thus exists in boolean algebras a universal relation† R such that

(1) *If $x R y$ and $y R z$, then $x R z$.*

I propose to find the most general boolean relation R which satisfies (1), and to show the connection between this relation and the syllogism relation $xy' = 0$.

I start with the fact that any universal relation between two boolean elements x, y is given by an equation of the form

$$(2) \qquad Axy + Bxy' + Cx'y + Dx'y' = 0.$$

Let (2) be a relation R satisfying (1). Then the *discriminants* A, B, C, D must be such that from

$$\text{(i)} \qquad Axy + Bxy' + Cx'y + Dx'y' = 0,$$
$$\text{(ii)} \qquad Ayz + Byz' + Cy'z + Dy'z' = 0,$$

we may conclude

$$\text{(iii)} \qquad Axz + Bxz' + Cx'z + Dx'z' = 0.$$

That is, the discriminants A, B, C, D must be such that equation (iii) is the result of eliminating y from equations (i) and (ii). Now (i) and (ii) together are equivalent to the single equation

$$\text{(iv)} \quad (Ax + Cx' + Az + Bz')y + (Bx + Dx' + Cz + Dz')y' = 0.$$

The result of eliminating y from (i) and (ii) is, then, the result

* Presented to the Society, September 7, 1923.

† That is, a relation given by a universal proposition.

of eliminating y from (iv). This result is

$$(Ax+Cx'+Az+Bz')\ (Bx+Dx'+Cz+Dz')=0,$$

or

$$A(B+C)xz+(B+A)\ (B+D)xz'$$
$$+(C+A)\ (C+D)x'z+(C+B)Dx'z'=0,$$

or

(v) $$A(B+C)xz+(B+AD)xz'$$
$$+(C+AD)x'z+D(B+C)x'z'=0.$$

Equating corresponding discriminants of (iii) and (v), we get, as necessary and sufficient conditions that (2) be a relation R satisfying (1),

$$A=A(B+C),\quad B=B+AD,\quad C=C+AD,\quad D=D(B+C),$$

or

$$AB'C'=0,\quad AB'D=0,\quad AC'D=0,\quad B'C'D=0,$$

or

(3a) $A+D<B+C$, and (3b) $AD<BC$.

We therefore find that the *most general boolean relation R which satisfies the proposition* (1) *is given by the relation*

(4) $$Axy+Bxy'+Cx'y+Dx'y'=0,$$
$$A+D<B+C,\quad AD<BC.$$

We find also that *the syllogism relation $xy'=0$ is the relation* (4) *in which* $A=C=D=0$, $B=1$.

We note that the relation of equivalence, which is $xy'+x'y=0$, is the relation (4), with $A=D=0$, $B=C=1$.

The statement that the relation (4) satisfies (1) is true even if the relation (4) be non-existent. Thus, the relation $xy+xy'+x'y+x'y'=0$, i. e., the relation $1=0$, is non-existent. But it satisfies (4) and it is true that *If* $1=0$ *and* $1=0$, *then* $1=0$. If we impose on (2) the condition that it be an *existent* relation, we must have

(5) $$ABCD=0.$$

Since condition (3b) can be written in the form

$$AD=AD\cdot BC,$$

conditions (3b) and (5) together are equivalent to

(6) $$AD=0,$$

and conditions (3*a*), (3*b*) become

$$A+D<B+C, \qquad AD=0. \tag{7}$$

Hence we find that *the most general existent boolean relation R which satisfies* (1) *is given by* (2) *and* (7).

Our main results may also be stated in the following form: *The totality of transitive universal relations in a boolean algebra is given by* (4). *The totality of existent transitive universal relations is given by*

$$Axy+Bxy'+Cx'y+Dx'y'=0, \quad A+D<B+C, \quad AD=0.$$

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ON CERTAIN QUINARY QUADRATIC FORMS*

BY E. T. BELL

1. *Introduction.* Except the classical theorems on the total number $N_5(n)$ of representations of the integer n as a sum of five integer squares, no explicit results on numbers of representations in quinary quadratic forms seem to have been obtained.† In general $N_5(n)$ is not expressible as a function of the real divisors alone of a single integer, but when n is a square, $N_5(n)$ is so expressible. This remarkable fact was found inductively by Stieltjes for $N_5(p^2)$, p prime, and proved for $N_5(n^2)$ by Hurwitz,‡ who showed that if $n=2^a m$, m odd, $N_5(n^2)=10\zeta_3(2^a)\,H(m)$, where

$$H(m)=[\zeta_3(p^a)-p\zeta_3(p^{a-1})]\,[\zeta_3(q^b)-q\zeta_3(q^{b-1})]\ldots,$$

$\zeta_r(n)$ being the sum of the rth powers of all the divisors of n, and $m=p^a q^b\ldots$ the prime factor resolution of m; by convention $H(1)=1$. In the course of his proof he showed that

$$\zeta_1(m^2)+2\zeta_1(m^2-2^2)+2\zeta_1(m^2-4^2)+\cdots=H(m).$$

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* Presented to the Society, December 27, 1923.

†Cf. Bachmann, *Zahlentheorie*, vol. 4, pp. 565–594.

‡COMPTES RENDUS, vol. 98 (1884), pp. 504–7; cf. Dickson's *History of the Theory of Numbers*, vol. 2, p. 311. For quadratic forms in $n>4$ variables, cf. ibid., vol. 3, chap. XI.

This identity makes it possible to find numerous theorems for the total number $N(2^\alpha m^2)$ of representations of $2^\alpha m^2$ in a quinary quadratic form other than a sum of five squares similar to Hurwitz' for $N_5(2^{2\alpha} m^2)$, viz., $N(n)$ is a function of the real divisors alone of a single integer when and only when* $n = 2^\alpha m^2$, m odd, $\alpha \geq 0$, and similarly for the number of proper representations. The results are so unexpectedly simple that we shall take space in § 3 to give 19 of them, or, counting the separate cases according to α, 72.

2. *Theorems on Representations.* Henceforth the variables x, y, z, t, u may take any integer values $\gtrless 0$; m, n, α, k_r are constant positive integers, m is odd, $n, \alpha \geqq 0$ are arbitrary, $k_r = 0$ when $r < 0$, and a, b, c, d, e are constant integers $\lesseqgtr 0$.

THEOREM I. *If the total number of representations of $2^\alpha m$ in the form $ax^2 + by^2 + cz^2 + dt^2$ is $k_\alpha \zeta_1(m)$, then the total number of representations of $2^\alpha m^2$ in the form $(a, b, c, d, \alpha) \equiv ax^2 + by^2 + cz^2 + dt^2 + 2^{\alpha+2} u^2$ is $k_\alpha H(m)$.*

Let $N'(n)$, $N(n)$ denote respectively the total numbers of representations of n in the forms $ax^2 + by^2 + cz^2 + dt^2$, $ax^2 + by^2 + cz^2 + dt^2 + eu^2$. Then, the sum continuing so long as the argument of N' is positive, evidently

$$N(n) = N'(n) + 2N'(n - e1^2) + 2N'(n - e2^2) + \cdots.$$

If $n = 2^\alpha m^2$, $e = 2^{\alpha+2}$, the argument of each N' is an odd multiple of 2^α. Hence

$$N(2^\alpha m^2) = k_\alpha[\zeta_1(m^2) + 2\zeta_1(m^2 - 2^2) + 2\zeta_1(m^2 - 4^2) + \cdots],$$

and the theorem follows by Hurwitz' identity.

As customary, a representation, in which the G. C. D. of x, y, z, t, u is 1 is called proper. To state the number of proper representations when the total number is given by Theorem I, we need $J_2(m)$, the Jordan totient of order 2 of m. If $m = \prod p_a$ is the resolution of m into powers of distinct primes > 1, $J_2(m) = m^2 \prod (1 - p^{-2})$, $J(1) = 1$ by convention, and $J_2(m)$ is the number of sets of two equal or distinct

* Because in each case it can be shown that $N(n) = cN_5(n)$, where c is a constant.

positive integers $\leq m$ whose G.C.D. is prime to m. Applying the general method of a former paper* to Theorem I, we find readily the following result.

THEOREM II. *If the total number of representations of $2^\alpha m^2$ in the form (a, b, c, d, α) is $k_\alpha H(m)$, the number of proper representations of $2^\alpha m^2$ in the same form is $(k_\alpha - k_{\alpha-2})m\, J_2(m)$.*

It is obvious that precisely similar considerations can be applied to any quadratic form in any number of variables, also to such forms when certain of the variables are restricted (examples in § 3 (2), (5)). The discussion is limited here to (a, b, c, d, α) because the application of the method to the deduction of explicit results for other forms presupposes a study of the general class of identities of which Hurwitz' is the second simplest example. The first such identity concerns the function which gives the number of representations of an integer as a sum of two squares. In both this and Hurwitz' identity only odd divisors enter; in the general case there appear only those divisors which are prime to the prime $p > 2$.

3. *Numbers of Representations.* From the way in which Theorems I, II are stated it is sufficient for any given form (a, b, c, d, α) to record the values of k_α and $k'_\alpha \equiv k_\alpha - k_{\alpha-2}$. The values of k_α are those which appertain to the quaternary form obtained from (a, b, c, d, α) by omitting the term in u. They can be found by referring to the papers of Liouville and Pepin.†

The results for quinary forms are given in the following list, in which the asterisk $(*)$ denotes any k with a suffix greater than that of the last k in the symbol in which the asterisk occurs. Thus (11) states that the total number

* ANNALS OF MATHEMATICS, (2), vol. 21 (1920), p. 170, § 8.

† JOURNAL DES MATHÉMATIQUES, (2), vol. 6 (1861), pp. 409, 440; vol. 7 (1862), pp. 1, 76, 80, 107, 112, 115, 119, 154, 156, 160, 163, 167; (4), vol. 6 (1890), pp. 5-67.

of representations of $2^\alpha m^2$ (m odd) in the form

$$x^2+2y^2+2z^2+4t^2+2^{\alpha+2}u^2$$

is 2, 4, 8, or 24 times $H(m)$ according as

$$\alpha = 0,\ 1,\ 2 \text{ or } > 2.$$

(1) $(1, 1, 1, 4, \alpha):[k_0, k_1, k_2, *] = [6, 12, 8, 24].$
(2) $(1, 1, 1, 4, \alpha),\ x+y$ odd$:k_1 = 8.$
(3) $(1, 1, 1, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24].$
(4) $(1, 1, 2, 2, \alpha):[k_0, k_1, *] = [4, 8, 24].$
(5) $(1, 1, 2, 2, \alpha),\ xy$ odd: $[k_1, k_2, *] = [4, 16, 0].$
(6) $(1, 1, 2, 8, \alpha):[k_1, k_2, k_3, *] = [6, 12, 8, 24].$
(7) $(1, 1, 4, 4, \alpha):[k_0, k_1, k_2, *] = [4, 4, 8, 24].$
(8) $(1, 1, 4, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24].$
(9) $(1, 1, 8, 8, \alpha):[k_1, k_2, k_3, *] = [4, 4, 8, 24].$
(10) $(1, 1, 16, 16, \alpha):[k_2, k_3, k_4, *] = [4, 4, 8, 24].$
(11) $(1, 2, 2, 4, \alpha):[k_0, k_1, k_2, *] = [2, 4, 8, 24].$
(12) $(1, 2, 2, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24].$
(13) $(1, 2, 4, 8, \alpha):[k_1, k_2, k_3, *] = [2, 4, 8, 24].$
(14) $(1, 2, 8, 16, \alpha):[k_2, k_3, k_4, *] = [2, 4, 8, 24].$
(15) $(1, 4, 4, 4, \alpha):[k_0, k_1, k_2, *] = [2, 0, 8, 24].$
(16) $(1, 4, 4, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24].$
(17) $(1, 4, 8, 8, \alpha):[k_1, k_2, k_3, *] = [0, 4, 8, 24].$
(18) $(1, 4, 16, 16, \alpha):[k_1, k_2, k_3, k_4, *] = [0, 4, 4, 8, 24].$
(19) $(1, 16, 16, 16, \alpha):[k_2, k_3, k_4, *] = [2, 0, 8, 24].$

When combined with the theorems on the number of representations of an integer as a sum of two and of three squares, all of these give interesting binary quadratic class number relations of a new kind, examples of which are presented in another paper (not yet published). To save space we shall omit the numbers of proper representations, which can be written down from the above by Theorem II; e. g., for the form (1) we get

$$(k'_0, k'_1, k'_2, k'_3, k'_4, *) = (6, 12, 2, 12, 16, 0).$$

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THE INVARIANTS OF FORMS UNDER THE BINARY LINEAR HOMOGENEOUS GROUP G_6 MODULO 2*

BY O. E. GLENN

1. *Introduction.* The transformation of an arbitrary binary quantic whose coefficients are written without binomial multipliers,

$$(1) \quad f = (a_0, \ldots, a_m)(x_1, x_2)^m = \prod_{i=0}^{m} (r_2^{(i)} x_1 - r_1^{(i)} x_2),$$

by the formulas

$$T: \ x_1 = \lambda_1 x_1' + \mu_1 x_2', \ x_2 = \lambda_2 x_1' + \mu_2 x_2',$$

in which λ_i, μ_i are such residues of a prime p that T ranges over the total group G of order $(p^2-p)(p^2-1)$, leads to the formal modular concomitant system of f. For various reasons $p = 2$ gives rise to exceptions in this theory; thus quadratic congruence theory becomes very special when $p = 2$, and also certain types of modular concomitants exist† for the even modulus that do not exist for $p > 2$.‡

2. *Analogies.* It is a known result of algebraic (non-modular) invariant theory that every concomitant of (1) is a polynomial in determinants of two types, viz. $(r^{(i)}r^{(k)})$, $(r^{(i)}x)$, i. e., linear forms themselves and resultants of pairs of them. Also the complete system of covariants of any number of quadratic quantics,

$$(2) \quad f_1 = (a_0, \ldots, a_2)(x_1, x_2)^2, \ldots, \ f_r = (l_0, \ldots, l_2)(x_1, x_2)^2,$$

is a set of concomitants that can be formed as transvectants of forms f_i taken in pairs.§ The dyadic combinations therefore furnish the complete seminvariant systems, also, but for the invariants it is found that the eliminants of the triadic combinations of the forms $f_1, \ldots, f_r$ are to be added.

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* Presented to the Society, December 28, 1923.

† TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 110.

‡ Dickson, *The Madison Colloquium Lectures*, Lecture III, p. 33-64.

§ Grace and Young, *Algebra of Invariants*, 1903, p. 161.

The facts are more complicated when the transformations form the group G_6 (mod 2). Then a simultaneous system of covariants of only two linear quantics

(3) $\qquad f = a_0x_1 + a_1x_2, \qquad g = b_0x_1 + b_1x_2,$

is composed of sixteen concomitants. These are tabulated below. The methods for their derivation are exemplified later in this paper in the corresponding problem for two quadratics although the two problems are not without differences in respect to detail. We shall use the abbreviations

(4) $$E_1 = a_0^2\frac{\partial}{\partial a_0} + a_1^2\frac{\partial}{\partial a_1},$$

$$E_2 = b_0^2\frac{\partial}{\partial b_0} + b_1^2\frac{\partial}{\partial b_1}, \qquad w = x_1^2\frac{\partial}{\partial x_1} + x_2^2\frac{\partial}{\partial x_2}$$

and, if two linear covariants are

$$C = c_0x_1 + c_1x_2, \quad D = d_0x_1 + d_1x_2,$$

$[CD]$ is the covariant

(5) $\qquad [CD] = (c_0d_0 + c_0d_1 + c_1d_0)x_1 + (c_1d_1 + c_0d_1 + c_1d_0)x_2.$

The covariant R in the list is led by the invariant (ab), viz.:

(6) $$R = (a_0b_1 + a_1b_0)x_1^3 + (a_0b_1 + a_0b_0 + a_1b_1)x_1^2x_2 + (a_1b_0 + a_0b_0 + a_1b_1)x_1x_2^2 + (a_0b_1 + a_1b_0)x_2^3.$$

Covariants of two linear forms under G_6:

Invariants: (ab), $\quad I = a_0^2 + a_0a_1 + a_1^2, \quad J = b_0^2 + b_0b_1 + b_1^2,$

$$L_1 = a_0^2a_1 + a_0a_1^2, \qquad L_2 = b_0^2b_1 + b_0b_1^2.$$

Linear Covariants: f, g, E_1f, E_2g, $[fg]$, $[gE_1f]$, $[fE_2G]$, $[E_1fE_2g]$.

Quadratic covariants: wf, wg, $Q = x_1^2 + x_1x_2 + x_2^2$.

Cubic covariants: R, $L = x_1^2x_2 + x_1x_2^2$.

Thus the independent covariants of a set of linear quantics, under G_6, form an extensive set which increases rapidly as the number of quantics is increased. If we add a third form $h = c_0x_1 + c_1x_2$ to the set f, g we shall have to consider concomitants formed from ground-forms in triadic combinations, as is proved by the existence of the following irreducible

seminvariant:

$$C = a_0b_0c_1 + a_0b_1c_1 + a_1b_0c_1 + a_1b_1c_0. \tag{7}$$

Similar considerations hold, under more complicated circumstances, for the simultaneous systems, for G_6, of a set of quadratic quantics. The rest of this paper concerns the system of two quadratics.

3. *Seminvariants.* Let Γ be the linear transformations upon a_0, a_1, a_2 which are induced by transforming f by

$$T\colon\ x = x_1' + x_2',\ x_2 = x_2',$$

and suppose Γ' to be that which corresponds by transformation of g by T, where

$$f = a_0x_1^2 + a_1x_1x_2 + a_2x_2^2,\quad g = b_0x_1^2 + b_1x_1x_2 + b_2x_2^2.$$

Then,

$$\begin{cases} \Gamma\colon\ a_0' = a_0,\ a_1' = a_1,\ a_2' = a_0 + a_1 + a_2, \\ \Gamma'\colon\ b_0' = b_0,\ b_1' = b_1,\ b_2' = b_0 + b_1 + b_2. \end{cases} \tag{8}$$

With suitable restrictions we connect the problem of the seminvariants of f and g with a simpler problem, previously solved, by taking $a_0 = 0$, temporarily. We then have two simultaneous groups (mod 2), viz.,

$$\begin{cases} \Gamma_1\colon\ a_1' = a_1,\ a_2' = a_1 + a_2, \\ \Gamma_1'\colon\ b_0' = b_0,\ b_1' = b_1,\ b_2' = b_0 + b_1 + b_2, \end{cases} \tag{9}$$

concerning which it is known that a fundamental system of universal concomitants consists of six quantics, as follows:*

$$a_1,\ b_0,\ b_1,\ \psi_1 = a_2^2 + a_1a_2,\ s = (b_0 + b_1 + b_2)b_2,\quad \varrho_1 = a_1b_2 + a_2(b_0 + b_1). \tag{10}$$

We desire six seminvariants of the set f, g such that the forms (10) are respectively residual to the six when $a_0 \equiv 0$ (mod 2).

The seminvariant of f of the type of s is $a_0a_2 + \psi_1 = \sigma$. The leading coefficient of $[fg]$† is $a_0b_0 + (fg)$, where (fg) is the invariant

$$(fg) = (a_0 + a_1)(b_1 + b_2) + (b_0 + b_1)(a_1 + a_2) + a_1b_1.$$

The form ϱ_1 is the residue (mod a_0) of $(fg) + (b_0 + b_1)a_1 = x$.

* TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 293.

† The symbolism (fg), $[fg]$, $\{fg\}$, $\{\overline{fg}\}$, explained in (19) in the present article, was first defined in PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 5 (1919), p. 107.

Assume, in arbitrary form, a seminvariant S of the set f, g,

$$S = S(a_0, a_1, a_2, b_0, b_1, b_2); \tag{11}$$

then, by (10),

$$S_1 = S(0, a_1, a_2, b_0, b_1, b_2) \equiv F(a_1, b_0, b_1, \psi_1, s, \varrho_1) \pmod 2, \tag{12}$$

where F is a polynomial in its arguments with integral coefficients. Hence we can arrange S as an expression of the form

$$S \equiv F(a_1, b_0, b_1, \sigma, s, x) + a_0 F_1(a_0, a_1, a_2, b_0, b_1, b_2) \pmod 2, \tag{13}$$

and F_1 is evidently a seminvariant of the set f, g. This process of reduction can be applied successively until we reach a coefficient quantic F_r which is free from a_0 (explicitly), it being, therefore, a polynomial in

$$a_1, b_0, b_1, \sigma, s, x,$$

i. e.,

$$S \equiv F + a_0 F_1 + a_0^2 F_2 + \cdots + a_0^r F_r \pmod 2. \tag{14}$$

THEOREM. *A fundamental system of seminvariants of the set f, g* (mod 2) *consists of the seven forms*

$$a_0, b_0, a_1, b_1, \sigma, s, (fg). \tag{15}$$

4. *Syzygies.* The expressions $k = a_0\sigma, \quad \varkappa = b_0 s, a_1, b_1, q_1 = \sigma + a_0^2 + a_0 a_1, \ q_2 = s + b_0^2 + b_0 b_1, (fg)$ are pure invariants. The following syzygies can be verified:

$$\begin{cases} a_0^3 + a_0^2 a_1 + a_0 q_1 + k = 0, \\ b_0^3 + b_0^2 b_1 + b_0 q_2 + \varkappa = 0, \\ \varrho^2 + (a_0 + a_1)(b_0 + b_1)\varrho + (a_0^2 + a_1^2)s + (b_0^2 + b_1^2)\sigma \\ \qquad + (a_0 b_0 + a_1 b_1)(a_0 b_1 + a_1 b_0) = 0, \end{cases} \tag{16}$$

where $\varrho = (fg) + a_1 b_1$.

These syzygies may be employed as literal moduli of reduction for the purpose of reducing the arbitrary seminvariant $S(a_0, a_1, a_2, b_0, b_1, b_2)$ to a polynomial of finite order in a_0, b_0. We have immediately the following theorem.

THEOREM. *The arbitrary seminvariant S of two quadratic forms f, g can be represented in the finite form*

$$S = \Phi_0 + \Phi_1 b_0 + \Phi_2 b_0^2 + a_0(\psi_0 + \psi_1 b_0 + \psi_2 b_0^2) + a_0^2(\chi_0 + \chi_1 b_0 + \chi_2 b_0^2), \tag{17}$$

in which Φ_i, ψ_i, χ_i *are polynomials in the invariants*

$$q_1,\ q_2,\ k,\ \varkappa,\ a_1,\ b_1,\ (fg), \tag{18}$$

and the highest power of (fg) *which occurs is the first.*

5. *A Method in Covariants.* It is known that a complete concomitant scale,* for the modulus 2, of a covariant,

$$C = C_0x_1^M + C_1x_1^{M-1}x_2 + \cdots + C_Mx_2^M,$$

of a quantic f_m of order m, for the reduction of all concomitants of degree unity in the coefficients of C and of order > 3, is composed of

$$\left\{\begin{array}{l} (C),\ (C) = C_1 + C_2 + \cdots + C_{M-1}, \\ \qquad\qquad [C] = (C_0 + (C))x_1 + ((C) + C_M)x_2, \\ \{C\} = C_0x_1^2 + (C)x_1x_2 + C_Mx_2^2, \\ \{\bar{C}\} = C_0x_1^3 + J_1x_1^2x_2 + J_2x_1x_2^2 + C_Mx_2^3, \\ \qquad\qquad (J_1 + J_2 \equiv (C) \pmod 2). \end{array}\right. \tag{19}$$

The latter covariant is existent only when M is an odd number. This scale produces concomitants of f_m from any covariant of the latter by the principle of copied forms.

Another method, not previously described, for the construction of covariants of f_m, to any prime modulus p, is to make an appropriate selection of a primary quantic,

$$P_0 = q_0x_1^\alpha + q_1x_1^{\alpha-1}x_2 + \cdots + q_\alpha x_2^\alpha, \tag{20}$$

of given degree-order (i, α) and apply to it, simultaneously, the substitutions of the group upon the variables generated by $x_1 = x_1' + x_2'$, $x_2 = x_2'$, and the corresponding substitutions of the induced group upon the coefficients. We thus obtain p quantics

$$\begin{aligned} &P_0,\ P_1 = (q_0', \ldots, q_\alpha')\,(x_1, x_2)^\alpha,\ \ldots, \\ &\qquad\qquad P_{p-1} = (q_0^{(p-1)}, \ldots, q_\alpha^{(p-1)})\,(x_1, x_2)^\alpha, \end{aligned} \tag{21}$$

and, if we assume that the primary quantic has been properly selected, any symmetric function of $P_0, P_1, \ldots, P_{p-1}$ is a covariant, modulo p, of f_m. Not many rules, other than em-

*Transactions of this Society, vol. 19 (1918), p. 110; vol. 20 (1919), p. 155.

pirical ones, for the determination of primary quantics, are known to the writer, but examples of the method are shown in the next paragraph. Obviously P_0 should be such that the assumed symmetric function is invariantive under the other two generators of the total group (mod p), i. e., $x_1 = x_1'$, $x_2 = \lambda x_2'$; $x_1 = x_2'$, $x_2 = -x_1'$.

6. *Fundamental Covariants.* We are evidently able to reduce all covariants in terms of those of orders 0, 1, 2, 3 led by seminvariants $a_0^i b_0^j$, $(i, j = 0, 1, 2)$ and by the invariants of which Φ_0 in (17) is a function. A formula showing this reduction in general form will be derived.

We find the following covariants with the leading coefficients which are adjoined. Abbreviations employed are

$$E_1 = a_0^2 \frac{\partial}{\partial a_0} + a_1^2 \frac{\partial}{\partial a_1} + a_2^2 \frac{\partial}{\partial a_2},$$

$$E_2 = b_0^2 \frac{\partial}{\partial b_0} + b_1^2 \frac{\partial}{\partial b_1} + b_2^2 \frac{\partial}{\partial b_2}.$$

Linear forms:

$a_0 + a_1, [f]$; $b_0 + b_1, [g]$; $a_0 b_0 + (fg)$, $[fg]$;
$a_0^2 + a_1^2, [E_1 f]$; $b_0^2 + b_1^2, [E_2 g]$; $a_0 b_0^2 + (f E_2 g)$, $[f E_2 g]$;
$a_0^2 b_0 + (g E_1 f), [g E_1 f]$; $a_0^2 b_0^2 + (E_1 f E_2 g)$, $[E_1 f E_2 g]$.

Quadratic forms:

a_0, f; b_0, g; $a_0^2, E_1 f$; $b_0^2, E_2 g$; $a_0 b_0$, $\{fg\}$;
$a_0^2 b_0$, $\{g E_1 f\}$; $a_0 b_0^2$, $\{f E_2 g\}$; $a_0^2 b_0^2$, $\{E_1 f E_2 g\}$.

There are no linear covariants, in the domain, led by invariants, and the only quadratic covariants whose leading coefficients are invariants are comprised in the formula IQ, where I is an arbitrary invariant. The only invariantive leader of covariants of the third order which we shall be required to consider is Φ_0 (cf. (17)). Let ζ be the operation of applying to a primary quantic P_0 the substitutions

(22) $$x_1 = x_1' + x_2', \quad x_2 = x_2'.$$
$$a_0' = a_0, \quad a_1' = a_1, \quad a_2' = a_0 + a_1 + a_2.$$

If the primary is

(23) $$P_0 = (a_0 + a_2)x_1 + (a_0 + a_1)x_2,$$

then

$$P_1 = \zeta P_0 = (a_1 + a_2)x_1 + (a_0 + a_2)x_2,$$

so that

$$(24)\quad \begin{cases} \sum P_0 = (a_0+a_1)x_1+(a_1+a_2)x_2 = [f], \\ \Delta_1 = \sum P_0P_1 = [(a_0+a_2)x_1+(a_0+a_1)x_2] \\ \qquad\qquad \times[(a_1+a_2)x_1+(a_0+a_2)x_2]. \end{cases}$$

If the primary quantic is

$$P_0 = (b_0+b_2)x_1+(b_0+b_1)x_2,$$

we obtain, similarly,

$$(25)\quad \begin{cases} \sum P_0 = (b_0+b_1)x_1+(b_1+b_2)x_2 = [g], \\ \Delta_2 = \sum P_0P_1 = [(b_0+b_2)x_1+(b_0+b_1)x_2] \\ \qquad\qquad \times[(b_1+b_2)x_1+(b_0+b_2)x_2]. \end{cases}$$

The respective seminvariant leading coefficients of the covariants $[f]\Delta_1$, $[g]\Delta_2$ are the invariants

$$(26)\qquad a_1q_1+k, \quad b_1q_2+\varkappa,$$

and these may replace k, $\varkappa$, respectively, in the system (18). Instead of (fg) in the fundamental system (15), we may employ $(fg)_1 = (fg)+a_1b_1$, which is the resultant of $[f]$ and $[g]$. A cubic covariant led by $(fg)_1$ is

$$(27)\qquad B = [fg]Q+[f]g+a_1[g]Q.$$

There exist no cubic covariants led by any of the invariants k, $\varkappa$, q_1, q_2, due to the fact that all of these invariants contain a term which is left unaltered by the permutational substitution* $(a_0a_2)(a_1)(b_0b_2)(b_1) = S_1$.

The cubic covariants which we require, with their leading coefficients, are listed below.

Cubic forms:

a_0+a_1, $[f]Q$; $a_0^2+a_1^2$, $[E_1f]Q$; b_0+b_1, $[g]Q$;
$b_0^2+b_1^2$, $[E_2g]Q$; $a_0b_0+(fg)$, $[fg]Q$; $a_0^2b_0+E_1(fg)$, $[E_1fg]Q$;
$a_0b_0^2+E_2(fg)$, $[fE_2g]Q$; $a_0^2b_0^2+E_1E_2(fg)$, $[E_1fE_2g]Q$; $(fg)_1$, B.

THEOREM. *An invariant leading coefficient Φ_0 of a cubic covariant of f, g is necessarily congruent modulo 2 to the expression*

$$(28)\qquad C = (a_1q_1+k)\psi_1'+(b_1q_2+\varkappa)\psi_2'+(fg)_1\psi_3',$$

where the quantics ψ_i' are invariants.

* TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 111.

To prove this we note that the invariant Φ_0 (cf. (17)) is always of the form

$$F(q_1, q_2, a_1, b_1) + C = \Phi_0, \tag{29}$$

where F is an integral form in its arguments. We have constructed covariants whose leaders are the invariants of the set

$$a_1 q_1 + k, \ b_1 q_2 + \varkappa, \ (fg)_1.$$

If G is a covariant whose leader is Φ_0, we have that

$$G + \psi_1'[f]\Delta_1 + \psi_2'[g]\Delta_2 + \psi_3 B \tag{30}$$

is a cubic covariant led by F, and this is an absurdity because every possible form F evidently contains* a term which is left unaltered by S_1. Hence $F \equiv 0 \pmod 2$, $\Phi_0 \equiv C \pmod 2$, which was to be proved.

7. *Reductions of the Arbitrary Linear, Quadratic, and Cubic Covariants.* Let $\sum_1$ represent a covariant of order unity whose seminvariant leading coefficient is S (cf. 17)). We have, identically, $S = I + J$, where

$$\left\{\begin{aligned} I = {} & (b_0 + b_1)\Phi_1 + (b_0^2 + b_1^2)\Phi_2 + (a_0 + a_1)\psi_0 \\ & + [a_0 b_0 + (fg)]\psi_1 + [a_0 b_0^2 + (fE_2 g)]\psi_2 + (a_0^2 + a_1^2)\chi_0 \\ & + [a_0^2 b_0 + (gE_1 f)]\chi_1 + [a_0^2 b_0^2 + (E_1 f E_2 g)]\chi_2; \\ J = {} & \Phi_0 + b_1\Phi_1 + b_1^2\Phi_2 + a_1\psi_0 + (fg)\psi_1 + (fE_2 g)\psi_2 \\ & + a_1^2\chi_0 + (gE_1 f)\chi_1 + (E_1 f E_2 g)\chi_2. \end{aligned}\right. \tag{31}$$

The following covariant is led by I:

$$\left\{\begin{aligned} K_1 = {} & \Phi_1[g] + \Phi_2[E_2 g] + \psi_0[f] + \psi_1[fg] + \psi_2[fE_2 g] \\ & + \chi_0[E_1 f] + \chi_1[gE_1 f] + \chi_2[E_1 f E_2 G]. \end{aligned}\right. \tag{32}$$

Therefore, there would exist a linear covariant led by the invariant J, viz., $\sum_1 + K_1$, unless $J \equiv 0 \pmod 2$. Thus every linear covariant $\sum_1$ is reduced by the formula (32), $(\sum_1 = K_1)$.

Let $\sum_2$ represent an arbitrary quadratic covariant which is led by S. The following covariant has S for seminvariant leader:

$$\left\{\begin{aligned} K_2 = {} & \Phi_0 Q + \Phi_1 g + \Phi_2 E_2 g + \psi_0 f + \psi_1\{fg\} \\ & + \psi_2\{fE_2 g\} + \chi_0 E_1 f + \chi_1\{gE_1 f\} + \chi_2\{E_1 f E_2 g\}. \end{aligned}\right. \tag{33}$$

Then K_2 differs from $\sum_2$ by some covariant which contains x_2

*The number $\binom{2m}{m}$ is even for all integers m.

as a factor; but, as no quadratic covariant can be factored thus, we have $\sum_2 \equiv K_2 \pmod 2$, i. e., the arbitrary quadratic covariant is reduced.

Let $\sum_3$ represent an arbitrary covariant of order 3 with seminvariant leader S. The following is a cubic covariant whose leading coefficient is I (cf. (31)): $K_3 = K_1 Q$. Note therefore that $\sum_3 + K_3 = \Gamma_3$ is a covariant whose leader is J. Hence, by (28) (Theorem),

$$J \equiv C + (fE_2g)_1\psi_2 + (gE_1f)_1\chi_1 + (E_1fE_2g)_1\chi_2,$$

that is,

$$\Gamma_3 = \psi_1'[f]\Delta_1 + \psi_2'[g]\Delta_2 + \psi_3'B + (\psi_2 E_2 + \chi_1 E_1 + \chi_2 E_1E_2)B, \tag{34}$$

where $(fE_2g)_1 = E_2(fg)_1 = (fE_2g) + a_1b_1^2$, (cf. (27)). Note that operations by E_1, E_2 upon B produce only polynomials in covariants already listed. We have now reduced $\sum_3$ to the form

$$\sum_3 = K_3 + \Gamma_3 + \Theta L \pmod 2, \tag{35}$$

since L is the only cubic covariant which contains x_2 as a factor. The quantic Θ is a pure invariant, but it is not known whether it is reducible, in all cases, entirely in terms of the invariants of the set (18). The following theorem has now been established.

THEOREM. *A fundamental system of formal covariants of the set consisting of the two binary quadratics*

$$f = (a_0, a_1, a_2)(x_1, x_2)^2, \quad g = (b_0, b_1, b_2)(x_1, x_2)^2, \tag{36}$$

under the total group G_6, modulo 2, is composed of 21 quantics, namely, seven invariants, $q_1, q_2, k, \varkappa, a_1, b_1, (fg)$, eight linear covariants, $[f], [g], [fg], [E_1f], [E_2g], [fE_2g], [gE_1f], [E_1fE_2g]$, five quadratic covariants, $Q, f, g, \Delta_1, \Delta_2$, and one cubic covariant, L.

The remaining forms are reducible, as follows:

$$E_1f = [f]^2 + a_1^2Q, \quad E_2g = [g]^2 + b_1^2Q,$$
$$\{fg\} = [f][g] + a_1g + b_1f + a_1b_1Q, \quad \{gE_1f\} = E_1\{fg\},$$
$$\{fE_2g\} = E_2\{fg\}, \quad \{E_1fE_2g\} = E_1E_2\{fg\}.$$

THE UNIVERSITY OF PENNSYLVANIA

ON THE APPLICATION OF THE THEORY OF IDEALS TO DIOPHANTINE ANALYSIS*

BY G. E. WAHLIN

1. *Introduction.* About three years ago† Professor Dickson stated a certain conjectured theorem, and he has recently published a proof of it.‡

After having examined a proof of the same theorem by the author of this article, Professor Dickson suggested the investigation of a more general equation than the one which he had considered, and the following pages contain the results of this investigation.

2. *Rings.* Let us consider any algebraic number field $k(\theta)$ of degree n. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be a fundamental system of integers of $k(\theta)$ so that every integer of the field can be represented by the fundamental form

$$x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n, \tag{1}$$

in which the $x_1, x_2, \ldots, x_n$ are rational integers.

By a *ring* R in $k(\theta)$ we understand a set of integers which is closed with respect to addition, subtraction, and multiplication, and which contains the rational integers. Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be a fundamental system of R. As above, we shall call

$$x_1\varrho_1 + x_2\varrho_2 + \cdots + x_n\varrho_n \tag{2}$$

the *fundamental form* of R. Every element of R is represented once and only once by (2) when the $x_1, x_2, \cdots, x_n$ are given rational integral values.

Since $\varrho_1, \varrho_2, \ldots, \varrho_n$ are integers in $k(\theta)$, they can be represented by (1), and we shall suppose that

$$\varrho_i = r_{i1}\gamma_1 + r_{i2}\gamma_2 + \cdots + r_{in}\gamma_n, \qquad (i = 1, 2, \ldots, n). \tag{3}$$

* Presented to the Society, December 29, 1923.

† L. E. Dickson, *A new method in Diophantine analysis*, this BULLETIN, vol. 27, No. 8 (May, 1921), p. 353.

‡ L. E. Dickson, *Integral solutions of* $x^2 - my^2 = zw$, this BULLETIN, vol. 29, No. 10 (Dec., 1923), p. 464.

The absolute value of the determinant $|r_{ij}|$ of the system (3) shall be called the *index of the ring*, and shall be denoted by Δ. Since any product of integers of R belongs to R the product

$$\varrho_1^{s_1} \cdot \varrho_2^{s_2} \cdots \varrho_n^{s_n}, \quad s_i \geqq 0, \qquad (i = 1, 2, \ldots, n)$$

can be represented by (2), and we shall write

$$(4) \quad \varrho_1^{s_1} \cdot \varrho_2^{s_2} \cdots \varrho_n^{s_n} = C^{(1)}_{s_1 s_2 \ldots s_n} \varrho_1 + C^{(2)}_{s_1 s_2 \ldots s_n} \varrho_2 + \cdots + C^{(n)}_{s_1 s_2 \ldots s_n} \varrho_n.$$

This equation uniquely defines the rational integers $C^{(i)}_{s_1 s_2 \ldots s_n}$.

Let us next consider any k integers $\alpha', \alpha'', \ldots, \alpha^k$ from R, and let

$$\alpha^{(i)} = a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n, \qquad (i = 1, 2, \ldots, k).$$

We may then write the product

$$(5) \qquad \alpha' \cdot \alpha'' \cdots \alpha^{(k)} = \sum B_{s_1 s_2 \ldots s_n} \varrho_1^{s_1} \cdot \varrho_2^{s_2} \cdots \varrho_n^{s_n},$$

where the summation extends over all terms for which $s_1 + s_2 + \cdots + s_n = k$, and

$$B_{s_1 s_2 \ldots s_n} = \sum a_{1 i_{11}} \cdots a_{1 i_{1 s_1}} \cdot a_{2 i_{21}} \cdots a_{2 i_{2 s_2}} \cdot \cdots \cdot a_{n i_{n 1}} \cdots a_{n i_{n s_n}}.$$

That is, $B_{s_1 s_2 \ldots s_n}$ is the sum of all possible products formed by taking s_1 elements from the first column, s_2 from the second, s_3 from the third, etc., and so chosen that no two elements belong to the same row in the matrix

$$m = \begin{Vmatrix} a_{11}, & a_{21}, & \ldots, & a_{n1} \\ a_{12}, & a_{22}, & \ldots, & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1k}, & a_{2k}, & \ldots, & a_{nk} \end{Vmatrix}.$$

If in (5) we now replace the power-products $\varrho_1^{s_1} \cdot \varrho_2^{s_2} \cdots \varrho_n^{s_n}$ by their expressions as furnished by (4), we have

$$(6) \qquad \alpha' \cdot \alpha'' \cdots \alpha^{(k)} = A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n,$$

where

$$(7) \qquad A_i = \sum B_{s_1 s_2 \ldots s_n} C^{(i)}_{s_1 s_2 \ldots s_n},$$

the summation extending over all $s_1, s_2, \ldots, s_n$ whose sum is k.

The A_i, for whose computation a definite process is thus given, are rational integers. They will be used in the application whose consideration is the object of this paper.

3. *Ideals in R.* The *conductor of a ring* is an ideal f in $k(\theta)$ such that the product of any integer of $k(\theta)$ by f belongs to R.* By a *ring ideal* we shall understand an ideal in R as defined by Bachmann.† That is, an ideal $I^{(R)}$ of R is a set of integers of R such that the sum and difference of any two integers of the set belong to the set; the product of any integer of the set by any integer of R belongs to the set; and the greatest common divisor of f and the moduli thus defined is the ring R.

Let $I^{(R)}$ be any ideal of R, and let $\beta_1, \beta_2, \ldots, \beta_n$ be a fundamental system of $I^{(R)}$. Since the β_i belong to R, they can be represented by the form (2). If we write

$$\beta_i = b_{i1}\varrho_1 + b_{i2}\varrho_2 + \cdots + b_{in}\varrho_n, \qquad (i = 1, 2, \ldots, n), \tag{8}$$

then the *norm of* $I^{(R)}$ in R, which we shall denote by $N_R(I^{(R)})$, is the absolute value of the determinant $|b_{ij}|$.‡

If $\alpha = e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n$ is an integer in $I^{(R)}$, and if we apply (8), we have $\alpha = a_1\varrho_1 + a_2\varrho_2 + \cdots + a_n\varrho_n$, where

$$a_i = b_{1i}e_1 + b_{2i}e_2 + \cdots + b_{ni}e_n, \qquad (i = 1, 2, \ldots, n), \tag{9}$$

whose matrix is the conjugate of that of (8).

In order to distinguish between ideals in R and ideals in $k(\theta)$, we shall speak of them as *ring ideals* and *field ideals*, respectively. Principal ring ideals will be designated by $[\lambda]$, where λ is an integer of R, and principal field ideals by $\{\lambda\}$, where λ is an integer of $k(\theta)$.

To every ideal $I^{(R)}$ of R there corresponds a field ideal I obtained by forming the product of $I^{(R)}$ and the unit ideal of $k(\theta)$; and if I is any ideal of $k(\theta)$ which is relatively prime to f, the numbers of I which belong to R constitute a ring ideal $I^{(R)}$ whose corresponding field ideal is I. The norm $N(I)$ of I is equal to the norm $N_R(I^{(R)})$ of $I^{(R)}$ in R.§ The index Δ of R is divisible by the conductor f of R.‖

* Bachmann, *Zahlentheorie*, p. 136.
† Bachmann, loc. cit., p. 363.
‡ Bachmann, loc. cit., chapter 2, p. 74.
§ Bachmann, loc. cit., chapter 9. No. 2.
‖ Bachmann, chapter 4, p. 136.

Two ideals $I_1^{(R)}$ and $I_2^{(R)}$ shall be called equivalent when R contains two integers α_1 and α_2 which are relatively prime to f, and $\alpha_2 I_1^{(R)} = \alpha_1 I_2^{(R)}$.* Equivalent ideals constitute a class.

For any given ring ideal $I^{(R)}$ there is an ideal $J^{(R)}$ such that the corresponding field ideal J is relatively prime to any given field ideal T and the product $I^{(R)} \cdot J^{(R)}$ is a principal ideal in R.

To prove this, we make use of the fact that an ideal $J_1^{(R)}$ exists, such that $J_1^{(R)}(I^{(R)} \cdot T_2^{(R)}) = [\lambda_1]$, a principal ideal in R.† Here T_2 is used to denote the product of the distinct prime factors of T which are relatively prime to f, and $T_2^{(R)}$ is used to denote the corresponding ring ideal.

Let T_1 be the product of the distinct prime factors of T which are divisors of f. Then $J_1^{(R)} \cdot T_2^{(R)}$ is relatively prime to T_1. For, since $J_1^{(R)}$ and $T_2^{(R)}$ are ideals in R, their product is an ideal in R. Hence this product is prime to f, and therefore also to T_1, which is a factor of f. There exists in $k(\theta)$ an ideal J_2 which is relatively prime to T_2, such that $J_2 \cdot I \cdot f = \{\lambda_2\}$.‡ Since λ_2 is divisible by f, it is an integer in R. I is the field ideal corresponding to the given ring ideal $I^{(R)}$.

Since λ_1 and λ_2 both belong to R, their sum $\lambda_1 + \lambda_2$ belongs to R, and since λ_1 and λ_2 are both divisible by $I^{(R)}$, there exists an ideal $J^{(R)}$ such that $I^{(R)} \cdot J^{(R)} = [\lambda_1 + \lambda_2]$.§

That λ_1 is divisible by $I^{(R)}$ is seen by its definition. Moreover, λ_2 is divisible by I and belongs to R and hence also to $I^{(R)}$; it is therefore divisible by $I^{(R)}$. Hence $\lambda_1 + \lambda_2$ is divisible by $I^{(R)}$.

The $J^{(R)}$ thus defined is such that the corresponding field ideal J is relatively prime to T. For, since $I^{(R)} \cdot J^{(R)} = [\lambda_1 + \lambda_2]$, by multiplying both members by the unit ideal of $k(\theta)$, we have $I \cdot J = \{\lambda_1 + \lambda_2\}$. Now λ_1 / I is relatively prime

* Bachmann uses the restricted equivalent in which $sgn N(\alpha_1) = sgn N(\alpha_2)$.

† Bachmann, loc. cit., p. 398.

‡ Bachmann, loc. cit., p. 221.

§ Bachmann, loc. cit., p. 369.

to T_1 and is divisible by T_2, and λ_2/I is relatively prime to T_2 and is divisible by T_1. Hence, if p is any prime factor of T and hence a factor of T_1 or T_2, either λ_1 or λ_2 is divisible by Ip, but not both. Hence their sum is not divisible by Ip, and therefore $\{\lambda_1+\lambda_2\}/I$ is not divisible by any prime factor of T. It follows that J is relatively prime to T. In the application to Diophantine equations this theorem will be used with $T=\{\Delta\}$.

4. *Decomposable Forms.* If $x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n$ is the fundamental form of the ideal $I^{(R)}$, then $N(x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n)$ is a form of degree n in the n variables $x_1, x_2, \ldots, x_n$, with rational integral coefficients. The highest common factor of the coefficient of this form is the norm of the ideal $I^{(R)}$ in R. Hence

$$(10)\quad N(x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n)=N_R(I^{(R)})F(x_1,x_2,\ldots,x_n),$$

where $F(x_1, x_2, \ldots, x_n)$ is a unit form decomposable in R.*

The following theorem is a modified form of one given by Bachmann.† The modifications are made so as to apply to ideals in R, and also for the equivalence as we have defined it.

If $J^{(R)}$ is any ideal of R and $F(x_1, x_2, \ldots, x_n)$ is a unit form obtained as above from an ideal $I^{(R)}$ of the class reciprocal to that of $J^{(R)}$, then there exist rational integers e_1, $e_2, \ldots, e_n$, such that $N_R(J^{(R)}) = |F(e_1, e_2, \ldots, e_n)|$; and, conversely, any rational integer $F(e_1, e_2, \ldots, e_n)$ represented by the decomposable form $F(x_1, x_2, \ldots, x_n)$ is in absolute value the norm of an ideal of the class reciprocal to that of $I^{(R)}$ provided $e_1\beta_1+e_2\beta_2+\cdots+e_n\beta_n$ is prime to f.

Let $\beta_1, \beta_2, \ldots, \beta_n$ be the fundamental system of $I^{(R)}$ from which the form $F(x_1, x_2, \ldots, x_n)$ is obtained, and let $I^{(R)}J^{(R)} = [\gamma]$. Then $\gamma = e_1\beta_1+e_2\beta_2+\cdots+e_n\beta_n$, and

$$\begin{aligned}|N(\gamma)| = N_R(I^{(R)})\cdot N_R(J^{(R)}) &= N(e_1\beta_1+e_2\beta_2+\cdots+e_n\beta_n)\\ &= N_R(I^{(R)})\,|F(e_1, e_2, \ldots, e_n)|,\end{aligned}$$

and hence

$$N_R(J^{(R)}) = |F(e_1, e_2, \ldots, e_n)|.$$

* Bachmann, loc. cit., chapter 10, No. 6.

† Bachmann, loc. cit., p. 429.

Conversely, if $F(e_1, e_2, \ldots, e_n)$ is any rational integer represented by $F(x_1, x_2, \ldots, x_n)$ corresponding to the ring ideal $I^{(R)}$, then

$$N_R(I^{(R)})F(e_1, e_2, \ldots, e_n) = N(e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n).$$

But if $e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n$ is prime to f, $[e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n]$ is a principal ideal in R; and, since it is divisible by $I^{(R)}$, there exists in the reciprocal class an ideal $J_1^{(R)}$ such that

$$J_1^{(R)} \cdot I^{(R)} = [e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n].$$

Hence we may write

$$|N(e_1\beta_1 + \cdots + e_n\beta_n)| = N_R(J_1^{(R)})\, N_R(I^{(R)}),$$

whence it follows that

$$N_R(J_1^{(R)}) = F(e_1, e_2, \ldots, e_n).$$

If $e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n$ is not prime to F and if I is the field ideal corresponding to $I^{(R)}$, then

$$\{\gamma\} = \{e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n\}$$

is divisible by I, and $\{\gamma\}/I = J_1$ is a field ideal which is not relatively prime to f. The method used above will show that the absolute value of $F(e_1, e_2, \ldots, e_n)$ is the norm of J_1.

5. *Application to Diophantine Equations.* We shall now turn our attention to the application of the foregoing facts regarding rings and ring ideals to the solution in rational integers $\xi_1, \xi_2, \ldots, \xi_n$; $u_1, u_2, \ldots, u_{k-2}$ of the equation

$$N(\xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n) = u_1 \cdot u_2 \cdots u_{k-2}, \tag{11}$$

where as before $\varrho_1, \varrho_2, \ldots, \varrho_n$ is a fundamental system of any ring R in $k(\theta)$.

Let us suppose that we have a set of integers satisfying (11). Since $\varrho_1, \varrho_2, \ldots, \varrho_n$ is a fundamental system of R, $\gamma = \xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n$ is an integer of $k(\theta)$. We shall suppose it resolved into its ideal prime factors. There are s distinct factors. We shall denote them by $p_1, p_2, \ldots, p_s$. Let us suppose that the first s_1 of these are relatively prime to f. Let $p_1^{\lambda_{11}}$ be the highest power of p_1 which is a factor of γ and whose norm is a factor of u_1; $p_1^{\lambda_{12}}$ the highest power of p_1 which is a factor of $\gamma/p_1^{\lambda_{11}}$ and

whose norm is a factor of u_2; $p_1^{\lambda_{12}}$ the highest power of p_1 which is a divisor of $\gamma/p_1^{\lambda_{11}+\lambda_{12}}$ and whose norm is a divisor of u_3; and so on, until finally $p_1^{\lambda_{1k-2}}$ is the highest power of p_1 which is a divisor of $\gamma/p_1^{\lambda_{11}+\lambda_{12}+\cdots+\lambda_{1k-3}}$ and whose norm is a factor of u_{k-2}.

We next consider the prime ideal p_2 in the same way, and we let $p_2^{\lambda_{21}}$ be the highest power of p_2 which is a divisor of γ and whose norm is a divisor of $u_1/N(p_1^{\lambda_{11}})$; $p_2^{\lambda_{22}}$ the highest power of p_2 which is a divisor of $\gamma/p_2^{\lambda_{21}}$ and whose norm is a divisor of $u_2/N(p_1^{\lambda_{12}})$; and we continue in this way until finally $p_{s_1}^{\lambda_{s_1k-2}}$ is the highest power of p_{s_1} which is a divisor of $\gamma/p_{s_1}^{\lambda_{s_11}+\lambda_{s_12}+\cdots+\lambda_{s_1k-3}}$ and whose norm is a divisor of $u_{k-2}/N(p_1^{\lambda_{1k-2}} \cdot p_2^{\lambda_{2k-2}} \cdots p_{s-1}^{\lambda_{s-1\,k-2}})$.

We shall now write

$$P_i = p_1^{\lambda_{1i}} \cdot p_2^{\lambda_{2i}} \cdots p_{s_1}^{\lambda_{s_1 i}}, \qquad (i = 1, 2, \ldots, k-2).$$

From the construction of the P_i, we see that γ is divisible by the product $P_1 \cdot P_2 \cdots P_{k-2}$, and that u_i is divisible by $N(P_i)$. We shall therefore write

$$Q = \frac{\{\gamma\}}{P_1 \cdot P_2 \cdots P_{k-2}},$$

and

$$u_i = \mu_i \cdot N(P_i).$$

Let us next write $Q = P_{k-1} \cdot P_k$ where P_k is the largest factor of Q all of whose prime divisors are divisors of f. We then have

$$\{\gamma\} = P_1 \cdot P_2 \cdots P_k,$$

where $P_1, P_2, \ldots, P_{k-1}$ are prime to f, and P_k contains no prime factor excepting factors of f. By (11), we have then

$$\begin{aligned} |N(\gamma)| &= N(P_1) \cdot N(P_2) \cdots N(P_k) = |u_1 \cdot u_2 \cdots u_{k-2}| \\ &= |\mu_1 \cdot \mu_2 \cdots \mu_{k-2}| \, N(P_1) \cdot N(P_2) \cdots N(P_{k-2}). \end{aligned}$$

Hence

$$N(P_{k-1}) \cdot N(P_k) = |\mu_1 \cdot \mu_2 \cdots \mu_{k-2}|.$$

Let us separate each μ_i into two factors μ_i' and μ_i'' such that

$$\begin{aligned} N(P_{k-1}) &= |\mu_1' \cdot \mu_2' \cdots \mu_{k-2}'|, \\ N(P_k) &= |\mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}''|. \end{aligned}$$

Since the $P_1, P_2, \ldots, P_{k-1}$, are all relatively prime to f, to each P_i, $(i<k)$, there corresponds a ring ideal whose norm in the ring is equal to the norm of P_i in $k(\theta)$. Let $P_i^{(R)}$ be the ring ideal corresponding to P_i. Let $I_i^{(R)}$ be an ideal from the reciprocal class. According to § 3, $I_i^{(R)}$ can be so chosen that the corresponding field ideal I_i is relatively prime to $\{\Delta\}$. We shall suppose that $I_i^{(R)}$ has been so chosen. Let $\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_n^{(i)}$ be a fundamental system of $I_i^{(R)}$, and let $F_i(x_1, x_2, \ldots, x_n)$ be the corresponding decomposable form. Then, by § 4, we have

$$N_R(P_i^{(R)}) = N(P_i) = \left| F_i(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}) \right|,$$

and hence

$$(12) \qquad u_i = \varepsilon_i \cdot \mu_i' \cdot \mu_i'' F_i(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}), \qquad (i = 1, 2, \ldots, k-2),$$

where ε_i is 1 or -1. Since $I_i^{(R)}$ and $P_i^{(R)}$ belong to reciprocal classes, $I_i^{(R)} \cdot P_i^{(R)} = [\alpha^{(i)}]$. Hence, since $[\alpha^{(i)}]$ is divisible by $I_i^{(R)}$, we have

$$\begin{aligned} \alpha^{(i)} &= e_1^{(i)} \beta_1^{(i)} + e_2^{(i)} \beta_2^{(i)} + \cdots + e_n^{(i)} \beta_n^{(i)} \\ &= a_{1i} \varrho_1 + a_{2i} \varrho_2 + \cdots + a_{ni} \varrho_n, \qquad (i = 1, 2, \ldots, k-1). \end{aligned}$$

Since P_k contains no prime factors except such as are factors of f, let us suppose that J is the smallest field ideal (i. e., the field ideal containing the fewest prime factors) whose product with P_k is divisible by f, and let M_1 be the smallest rational integer which is divisible by J. Let $M_2 = N(I)$, where $I = I_1 \cdot I_2 \cdots I_{k-1}$.

The ideal I is relatively prime to the principal ideal $\{\Delta\}$. Hence, since Δ is a rational integer, M_2 and Δ are relatively prime. Since M_2 is divisible by I, we can choose I_k such that $I \cdot I_k = \{M_2\}$, and I_k is then relatively prime to Δ, and hence also to f, which is a divisor of Δ.

Since $\{\gamma\} = P_1 \cdot P_2 \cdots P_k$, P_k belongs to the class reciprocal to that of $P_1 \cdot P_2 \cdots P_{k-1}$. But I belongs to the class reciprocal to that of $P_1 \cdot P_2 \cdots P_{k-1}$. Hence P_k and I belong to the same class, and P_k and I_k belong to reciprocal classes. Therefore $P_k I_k = \{\bar{\alpha}^{(k)}\}$.

Let us now write $\bar{\alpha} = \alpha' \cdot \alpha'' \cdots \alpha^{(k-1)} \cdot \bar{\alpha}^{(k)} \cdot M_1$. Since $\bar{\alpha}^{(k)} M_1$ is divisible by JP_k, it is divisible by f, and hence $\bar{\alpha}$ is an integer in R. In fact, $\bar{\alpha}^{(k)} \cdot M_1$ belongs to R, and we may therefore write

$$\bar{\alpha}^{(k)} \cdot M_1 = \bar{a}_{1k}\varrho_1 + \bar{a}_{2k}\varrho_2 + \cdots + \bar{a}_{nk}\varrho_n.$$

Moreover

$$\{\bar{\alpha}\} = P_1 \cdot P_2 \cdots P_k \cdot I_1 \cdot I_2 \cdots I_k \cdot M_1 = \{\gamma M_2 M_1\},$$

and hence $\bar{\alpha}$ and $\gamma M_1 M_2$ differ only by a factor which is a unit in $k(\theta)$. Then let us write $\bar{\alpha} = E\gamma \cdot M_1 \cdot M_2$. But since $\bar{\alpha}^{(k)} M_1$ is divisible by f, $E \cdot \bar{\alpha}^{(k)} \cdot M_1$ is also divisible by f, and hence belongs to R. Then, if we put $E\bar{\alpha}^{(k)} = \alpha^{(k)}$, we may write

$$M_1 E \bar{\alpha}^{(k)} = M_1 \alpha^{(k)} = a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n;$$

and since $\{\bar{\alpha}^{(k)}\} = \{\alpha^{(k)}\}$, we have $P_k I_k = \{\alpha^{(k)}\}$, and

$$\alpha = \alpha' \cdot \alpha'' \cdots \alpha^{(k)} M_1 = \gamma \cdot M_1 \cdot M_2.$$

We therefore have

$$M_1 M_2 \gamma = M_1 M_2 (\xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n)$$

$$= \prod_{i=1}^{n} (a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n),$$

and using the notations of § 2, we have

$$\xi_i = \frac{A_i}{M_1 \cdot M_2}, \qquad (i = 1, 2, \ldots, n). \tag{13}$$

Here the A_i are polynomials in the a_{ji}, $(j = 1, 2, \ldots, n;$ $i = 1, 2, \ldots, k)$, which upon application of (9) to the elements standing in the first $k-2$ rows of the matrix of § 2 as they occur in the A_i, gives an expression for the ξ_i in terms of the parameters $e_j^{(i)}$ which occur in the expressions for u_i, and the elements of the last two rows of the matrix which are implicitely involved in the u_i in the factors μ_i' and μ_i''.

We have thus by means of (12) and (13) obtained a general form for the solution of (11). We shall next see that all such expressions, when the parameters are given rational integral values, constitute a solution of (11).

Substituting for $\xi_1, \xi_2, \ldots, \xi_n$; $u_1, u_2, \ldots, u_{k-2}$ their values

as given by (12) and (13) in (11), we have the equation

$$(14)\qquad \frac{N(A_1\varrho_1+\cdots+A_n\varrho_n)}{M_1^n\cdot M_2^n}=\prod_{i=1}^{k-2}\varepsilon_i\mu_i'\mu_i''F_i(e_1^{(i)},e_2^{(i)},\ldots,e_n^{(i)}).$$

Since

$$M_2=I\cdot I_k,\ N(I_i)=N_R(I_i^{(R)}),\ |\mu_1'\cdot\mu_2'\cdots\mu_{k-2}'|=N_R(P_{k-1}^{(R)}),$$

and $P_{k-1}^{(R)}\cdot I_{k-1}^{(R)}=[\alpha^{(k-1)}]$, a principal ideal in R, we can write (14) in the form

$$(15)\qquad N(A_1\varrho_1+A_2\varrho_2+\cdots+A_n\varrho_n)$$
$$=M_1^nM_2^{n-1}N_R(I_{k-1}^{(R)})\mu_1'\cdot\mu_2'\cdots\mu_{k-2}'\prod_{i=1}^{k-2}\varepsilon_i\mu_i''N_R(I_i^{(R)})\times F_i(e_1^{(i)},\ldots,e_n^{(i)}).$$

But

$$N_R(I_i^{(R)})F_i(e_1^{(i)},e_2^{(i)},\ldots,e_n^{(i)})=N(e_1^{(i)}\beta_1^{(i)}+\cdots+e_n^{(i)}\beta_n^{(i)})$$
$$=N(a_{1i}\varrho_1+a_{2i}\varrho_2+\cdots+a_{ni}\varrho_n).$$

Also

$$N_R(I_{k-1}^{(R)})\cdot|\mu_1'\cdot\mu_2'\cdots\mu_{k-2}'|=N_R(I_{k-1}^{(R)})N_R(P_{k-1}^{(R)})$$
$$=N\{\alpha^{(k-1)}\}=\varepsilon^{k-1}N(\alpha^{(k-1)})$$
$$=\varepsilon_{k-1}N(a_{1k-1}\varrho_1+a_{2k-1}\varrho_2+\cdots+a_{nk-1}\varrho_n).$$

Let us suppose that the signs of $\mu_1', \mu_2', \ldots, \mu_{k-2}'$ have been so chosen that the sign of their product is the same as the sign of $N(\alpha^{(k-1)})$ and hence $\varepsilon_{k-1}=+1$. If we now put $\varepsilon=\varepsilon_1\cdot\varepsilon_2\cdots\varepsilon_{k-2}$, we may write (15) in the form

$$(16)\qquad N(A_1\varrho_1+A_2\varrho_2+\cdots+A_n\varrho_n)$$
$$=\varepsilon M_1^nM_2^{n-1}\mu_1''\cdot\mu_2''\cdots\mu_{k-2}''\prod_{i=1}^{k-1}N(a_{1i}\varrho_1+\cdots+a_{ni}\varrho_n).$$

Since $M_2=I\cdot I_k=N(I)$, we have

$$M_2^n=N(I)N(I_k)=M_2N(I_k),$$

and hence $M_2^{n-1}=N(I_k)$. Also $|\mu_1''\cdot\mu_2''\cdots\mu_{k-2}''|=N(P_k)$, and therefore

$$(17)\qquad M_1^nM_2^{n-1}|\mu_1''\cdot\mu_2''\cdots\mu_{k-2}''|=M_1^nN(I_k\cdot P_k)=N\{M_1\alpha^{(k)}\}$$
$$=N\{a_{1k}\varrho_1+a_{2k}\varrho_2+\cdots+a_{nk}\varrho_n\}.$$

For, since $M_1\alpha^{(k)}$ is divisible by f, it belongs to R. As above, we shall now affix signs to the μ_i'' so that the sign

of their product is the same as the sign of $N(\alpha^{(k)})$. Then we have

$$M_1^n \cdot M_2^{n-1} \mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}'' = N(a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n).$$

We may now write (16) in the form

$$N(A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n) = \varepsilon \prod_{i=1}^{k} N(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n),$$

which, by § 1, is seen to be an identity if $\varepsilon = +1$.

Hence, with the signs of the μ_i' and μ_i'' properly chosen, and ε_i so chosen that their product is $+1$, all numbers obtained by (12) and (13) are solutions of (11). We observe, however, that the expressions for $\xi_1, \xi_2, \ldots, \xi_n$ are fractional in form. Hence we must next determine under what conditions they are integral, that is, we must determine what conditions must be imposed on the μ_i' and μ_i'' in order that the solutions shall be integral, or in other words, that γ shall belong to R.

From the development, it follows that, for

$$\gamma = \frac{A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n}{M_1 M_2},$$

$M_1 \cdot M_2 \cdot \gamma$ is an integer in R and

$$\{\gamma\} = \left\{\frac{A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n}{M_1 M_2}\right\} = P_1 \cdot P_2 \cdots P_k$$

is a principal ideal of the field. Since the product of any integer of the field by Δ is an integer of R, we know that

$$\frac{A_1\Delta\varrho_1 + A_2\Delta\varrho_2 + \cdots + A_n\Delta\varrho_n}{M_1 M_2}$$

is an integer of R. Hence it is equal to $C_1\varrho_1 + C_2\varrho_2 + \cdots + C_n\varrho_n$, where $C_1, C_2, \ldots, C_n$ are rational integers. But the representation by the fundamental system is unique; hence

$$\frac{A_i\Delta}{M_1 M_2} = C_i.$$

Since M_2 is relatively prime to Δ, this says that A_i is divisible by M_2; hence (13) will not give fractions with factors of M_2 in the denominator.

We have defined J as the smallest ideal whose product with P_k is divisible by f, and M_1 as the smallest rational integer divisible by J. Let $M_1 = M_1' \cdot M_1''$, where M_1' is the smallest factor of M_1 such that $M_1' P_k I_k = \{\gamma^{(k)}\}$, when $\gamma^{(k)}$ is an integer of R. If we then choose $\alpha^{(k)}$ in $P_k \cdot I_k = \{\alpha^{(k)}\}$ such that $M_1 \alpha^{(k)} = M_1'' \gamma^{(k)}$, which is always possible since

$$M_1''\{\gamma^{(k)}\} = M_1 P_k I_k = M_1' M_1''\{\alpha^{(k)}\},$$

we see that the $a_{1k}, a_{2k}, \ldots, a_{nk}$ are all multiples of M_1'' since $\gamma^{(k)}$ belongs to R. We thus see that the A_i are all divisible by M_1'' and only factors of M_1' can occur in the denominators of the numbers furnished by (13).

In the first part of this article, we have seen that all integral solutions of (11) can be expressed by (12) and (13). In the proof as given, for $i < k$ the

$$e_1^{(i)}\beta_1^{(i)} + e_2^{(i)}\beta_2^{(i)} + \cdots + e_n^{(i)}\beta_n^{(i)} = a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n$$

were all relatively prime to f. Hence the product

$$\prod = \prod_{i=1}^{k-1}(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n)$$

is also relatively prime to f. The integer $\prod$ belongs to R. Hence if

$$(a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n)\prod = \prod_{i=1}^{k}(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \ldots + a_{ni}\varrho_n)$$

should have all its coefficients, when it is written as a linear function of $\varrho_1, \varrho_2, \ldots, \varrho_n$, divisible by some factor $M_1^{(3)}$ of $M_1' = M_1^{(3)} \cdot M_1^{(4)}$ it would follow that

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)} \cdot \prod$$

would be an integer of R.

Hence the product of

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)}$$

by $\prod$, or by any integer divisible by f, would be an integer of R. But $\prod$ belongs to R, and the principal ideal $[\prod]$

is relatively prime to f. Hence there exists an integer C in R and an integer D in f such that $C \cdot \prod + D = 1$.*

Since $C\prod$ belongs to R, the products of

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_k}{M_1' \cdot M_1''} M_1^{(4)}$$

by $C\prod$ and by D, and hence also the sum of these products, belong to R. Therefore

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)}$$

is an integer in R. But $\{\alpha^{(k)}\} = P_k I_k$, and

$$M_1' \cdot M_1'' P_k I_k = \{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n\}.$$

Hence

$$M_1^{(4)} P_k I_k = \left\{ \frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)} \right\},$$

where the integer determining the principal ideal belongs to R. But we have assumed that M_1' is the smallest factor of M_1 such that when $M_1' P_k I_k = \{\gamma^{(k)}\}$ the $\gamma^{(k)}$ belongs to R; hence $M_1^{(4)} = M_1'$, i. e., $M_1^{(3)} = 1$.

Therefore M_1' will always occur as a denominator in the numbers furnished by (13), and integral solutions are possible only when $M_1' = 1$. Consequently, in order to have integral solutions, $\alpha^{(k)}$ must be an integer of R.

We have seen that $a_{1k}, a_{2k}, \ldots, a_{nk}$ are all divisible by M_1''. Hence if we put $a_{ik}/M_1'' = C_{ik}$ we shall have

$$\gamma^{(k)} = C_{1k}\varrho_1 + C_{2k}\varrho_2 + \cdots + C_{nk}\varrho_n.$$

The matrix obtained from m in § 2 by replacing the a_{ik} by the C_{ik}, $(i = 1, 2, \ldots, n)$, shall be denoted by m'.

From the theory of the correspondence between ideals and decomposable forms, we know that to a class of ideals corresponds a class of forms. Any form of the class can be obtained by proper choice of the base of the ideal, from any ideal of the corresponding class of ideals by the method of § 4. Moreover, the numbers represented by any form of the class can be represented by every form of the class.

Let γ be any integer of R. Suppose that the principal field ideal $\{\gamma\}$ is separated into the product $T \cdot P_k$ of two field

* Bachmann, loc. cit., p. 367.

ideals such that T is relatively prime to f, and that P_k contains no prime factors except such as are divisors of f. Let $T^{(R)}$ be the ring ideal corresponding to T. From the reciprocal class in R, select an ideal $I^{(R)}$ whose corresponding field ideal I is relatively prime to $\{\Delta\}$. Let $N(I) = M_2$. Then M_2 is relatively prime to Δ, and hence also to f, which is a divisor of Δ. Therefore $I_k = \{M_2\}/I$ is relatively prime to f, and the corresponding ring ideal $I_k^{(R)}$ belongs to the class reciprocal to that of $I^{(R)}$.

Since $I^{(R)}$ and $T^{(R)}$ belong to reciprocal classes in R, we have $I^{(R)}T^{(R)} = [\alpha]$. Multiplying both members of this equation by the unit ideal of $k(\theta)$, we have $I \cdot T = \{\alpha\}$. It is easily seen that I_k and P_k belong to reciprocal classes. Hence $I_k \cdot P_k = \{\gamma^{(k)}\}$. We shall next see that the integer $\gamma^{(k)}$ belongs to R.

We have $T \cdot P_k = \{\gamma\}$ and γ was chosen an integer of R. Since M_2 is a rational integer, $M_2\gamma$ belongs to R. Since $I^{(R)}$ was chosen from the class reciprocal to that of $T^{(R)}$ in R, α belongs to R and is relatively prime to f. We may write

$$I_k P_k = \frac{\{M_2\}}{I} \cdot P_k = \frac{\{M_2\} T \cdot P_k}{I \cdot T} = \left\{\frac{M_2\gamma}{\alpha}\right\} = \{\gamma^{(k)}\},$$

and hence we may write $\gamma^{(k)} = M_2\gamma/\alpha$. Therefore $\gamma^{(k)} \cdot \alpha$ belongs to R; and, if a is any integer of $[\alpha]$, $\gamma^{(k)} \cdot a$ belongs to R. Also if b is any integer of f, $\gamma^{(k)} \cdot b$ belongs to R and therefore $\gamma^{(k)}(a+b)$ belongs to R. But since $[\alpha]$ is relatively prime to f, a and b may be so chosen that $a + b = 1$. Hence $\gamma^{(k)}$ belongs to R.

We may now sum up the result of the investigation as follows. Let $k(\theta)$ be any algebraic number field of degree n, and $\varrho_1, \varrho_2, \ldots, \varrho_n$ a fundamental system of a ring R whose conductor is f and index Δ. Select any integer γ from R and separate the principal ideal $\{\gamma\}$ into two factors $T \cdot P_k$, where T is relatively prime to the conductor f, and where P_k contains no prime factors except divisors of f. Let $T^{(R)}$ be the ring ideal corresponding to T, and $I^{(R)}$ an ideal from the reciprocal class in R whose corresponding field ideal I is relatively prime to $\{\Delta\}$. Let $M_2 = N(I)$, and $I_k = M_2/I$.

Then

$$I_k \cdot P_k = \{\gamma^{(k)}\} = \{C_{1k}\varrho_1 + C_{2k}\varrho_2 + \cdots + C_{nk}\varrho_n\},$$

since, as we have seen above, $\gamma^{(k)}$ belongs to R.

Select $k-2$ rational integers $\mu_1'', \mu_2'', \ldots, \mu_{k-2}''$, such that their product has the same sign as $N(\gamma^{(k)})$, and an absolute value equal to $N(P_k)$.

Next select $k-1$ ideals $I_1, I_2, \ldots, I_{k-1}$ whose product is I. As before, let $I_i^{(R)}$ be the ring ideal corresponding to I_i. Let $F_i(x_1, x_2, \ldots, x_n)$, $(i = 1, 2, \ldots, k-2)$, be the decomposable forms corresponding to the ideals $I_i^{(R)}$, $(i = 1, 2, \ldots, k-2)$. Choose an ideal $P_{k-1}^{(R)}$ from the class reciprocal to that of $I_{k-1}^{(R)}$ and let

$$P_{k-1}^{(R)} \cdot I_{k-1}^{(R)} = [\alpha^{(k-1)}] = [a_{1k-1}\varrho_1 + a_{2k-1}\varrho_2 + \cdots + a_{nk-1}\varrho_n].$$

Next select $k-2$ rational integers $\mu_1, \mu_2', \ldots, \mu_{k-2}'$ whose product has the sign of $N(\alpha^{(k-1)})$ and the absolute value $N(P_{k-1})$. Then for rational integral $e_j^{(i)}$, $(i = 1, 2, \ldots, k-2;$ $j = 1, 2, \ldots, n)$, and $\epsilon_i = +1$ or -1, such that $\epsilon_1 \cdot \epsilon_2 \cdots \epsilon_{k-2}$ $= +1$ the numbers

$$u_i = \epsilon_i \mu_i' \cdot \mu_i'' F_i(e_1^{(i)} \cdot e_2^{(i)} \cdots e_n^{(i)}), \qquad (i = 1, 2, \cdots, k-2),$$

$$\xi_i = \frac{A_i}{M_2}, \qquad (i = 1, 2, \cdots, n),$$

constitute a solution of the equation

$$N(\xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n) = u_1 \cdot u_2 \cdots u_{k-2}.$$

The A_i are computed as in § 2 from the matrix

$$m' = \begin{Vmatrix} a_{11}, & a_{21}, & \ldots, & a_{n1} \\ a_{12}, & a_{22}, & \ldots, & a_{n2} \\ & \cdots & \cdots & \\ a_{1k-1}, & a_{2k-1}, & \ldots, & a_{nk-1} \\ c_{1k}, & c_{2k}, & \ldots, & c_{nk} \end{Vmatrix}$$

when the a_{ij} in the first $k-2$ rows of the matrix are obtained from the $e_j^{(i)}$ by means of $k-2$ sets of equations such as (9), one set for each of the ideals $I_i^{(R)}$, $(i = 1, 2, \ldots, k-2)$.

By the same method, all solutions of the given Diophantine equation may be obtained.

THE UNIVERSITY OF ILLINOIS

ARTHUR DUNN PITCHER
IN MEMORIAM*

Arthur Dunn Pitcher, Professor of Mathematics in Adelbert College, Western Reserve University, died on October 5, aged forty-three years. While he had suffered from a weak heart for some time, his sudden death came as an unexpected blow to his family and colleagues. In fact he had met his classes as usual on October 4, and gave every indication of normal health.

He was born and reared in Kansas. He was graduated from the University of Kansas in 1906 and received the degree of Doctor of Philosophy from the University of Chicago in 1910. After that he held positions at the University of Kansas, Dartmouth College, and during the past eight years at Western Reserve University.

At the University of Chicago Professor Pitcher came under the influence of Professor E. H. Moore, under whose direction he wrote his doctor's thesis. After receiving his doctorate he continued the researches he had begun at Chicago in E. H. Moore's *General Analysis*; and these led him into researches in the Fréchet Theory and the Theory of Functions of a Real Variable. By his contributions to these fields of mathematics he attained a high place among American mathematicians.

While his principal interest was in the study of mathematics, nevertheless he performed conscientiously all the tasks that came to him. For the last three years he was a member of the budget committee and of the executive committee of Adelbert College. His work on these important committees gained for him the reputation of being fair minded and having sound judgment. As a consequence he was frequently consulted on matters of university policy.

At all times he maintained a high standard, but because of his patience and sympathy he enjoyed to an unusual degree the loyalty and affection of his students. The attainments of his students and their love for him called forth the admiration of his associates. His happy disposition and his kindly helpfulness to the students, to his associates, and to his friends, are sadly missed.

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* This statement was adopted as a minute by the Faculty of Western Reserve University, and is here published by permission.

EVANS' COLLOQUIUM LECTURES

The Cambridge Colloquium Lectures, 1916. *Part* I: *Functionals and their Applications. Selected Topics, including Integral Equations.* By G. C. Evans. New York, American Mathematical Society, 1918. 136 pp.

The theory of functionals has been in the foreground so much in the last two decades, particularly in the domain of integral equations, that a colloquium on this subject was very timely. The theory of integral equations, however, bears the same relation to the whole theory of functionals as the theory of algebraic equations does to the entire domain of the theory of functions. In fact, so rapid has been the development of the theory of functionals that Paul Lévy has recently devoted a book of nearly four hundred fifty pages to the study of functional analysis alone, leaving aside entirely the theory of complex functionals and touching but slightly the theory of integral and integro-differential equations and a number of other subjects that would ordinarily be included as topics in the theory of functionals. It can be seen from this what a difficult, and what seems to the reviewer an almost impossible, task it would be to essay a clear presentation of almost every domain of this difficult subject in the space of less than one hundred fifty pages. This was the task that Professor Evans set himself.

The author has divided his work into five lectures. The first lecture takes up general considerations of a functional, such as definitions of continuity, Volterra derivatives, and additive and non-additive functionals of plane curves. A rather interesting connection is noted between additive functionals and functions of point sets, viz., "an additive continuous functional of finite variation has a finite derivative (in the restricted sense) at all points except possibly those of a set of measure zero". Another interesting result is the extension of the law of the mean of the differential calculus. Functionals of space curves are then introduced, and the concept of the *flux of a functional,* due to Lévy, is defined, and is used extensively in the next lecture.

In studying the dependence of the Green's function on the boundary of the closed curve, Hadamard was led to an equation involving functional derivatives. These equations are the analogs of total and partial differential equations in n variables. Just as, in the latter, certain conditions of integrability are required, so here also analogous conditions must be satisfied. These are developed very elegantly by means of another concept due to Lévy, that of the *adjoint linear functional.* Various interesting applications are made of the integrability conditions, and, in particular, it is shown that Hadamard's equation is *completely integrable.*

The author now turns his attention to complex functionals, to which he devotes Lecture II. As far back as 1889 a considerable portion of this theory was developed by Volterra, who takes as a starting point the relation of *isogeneity*, which is the extension to functionals of curves in space of the relation that holds between two complex point functions on a surface. This condition of isogeneity is expressed in terms of the normal component to the curve of the vector flux of a functional defined in Lecture I. Although the author summarizes the properties of the linear vector function used in this lecture, it would be highly desirable for one to have some acquaintance with vector analysis before attempting to read it. In case the functionals involved are additive, the condition of isogeneity may be considerably simplified, since in this case the vector flux can be chosen as a vector point function independent of the curve. Important properties of the relation of isogeneity for additive functionals are given, and the analogs of Green's and Cauchy's theorems are proved. All this has been extended by Volterra to additive complex functionals whose arguments are r-dimensional hyperspaces immersed in n-space. For $r=n-2$, this is a direct generalization of the theory of complex functionals of curves in 3-space.

Although Lecture III has the caption "Implicit Functional Equations", this part of the work is essentially concerned with the study of the linear functional. This study must necessarily precede any existence theorem on implicit functional equations, since the differential, which has a natural origin here, is a linear functional. $T[\Phi]$ is said to be a *linear functional* of Φ if (1) it is a distributive functional of Φ, and (2) if it is a continuous functional of Φ, where Φ is allowed to range over the whole class of continuous functions. Various representations of such a linear functional have been given by Hadamard, F. Riesz, and Lebesgue, using as their means of representation the Riemann, the Stieltjes, and the Lebesgue integrals, respectively. Since, in the definition of a linear functional, one may change either the class of functions Φ, or the type of continuity, one may obtain two distinct extensions of the linear functionals. The first has been made by Fréchet, Lebesgue, Radon, and others. The second extension is to introduce higher orders of continuity. The importance of this concept seems to have been recognized first by Bliss,* who notes that if the calculus of variations is to be regarded as a chapter in the theory of maxima and minima of functionals, the definition of continuity of a functional must be extended in this way.

Fréchet's definition of a differential of a functional is now given, and a theorem on implicit functional equations due to Volterra is proved

* *A note on functions of lines,* PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 1 (1915), p. 173.

in which the differential is used. It must be pointed out, however, that Volterra uses a very special kind of differential. In fact, Volterra proved his result some time before Fréchet gave his formal definition.

Lecture IV opens with the remark that the most desirable approach to the treatment of Laplace's equation is to study, as Bôcher did, the equation

$$\int_c \frac{\partial u(x, y)}{\partial n}\, ds = 0.$$

This relation is supposed to hold for all circles within a given two-dimensional region, and the existence and continuity of u, u_x, u_y are assumed, instead of the existence of u_{xx}, u_{yy}, which have no physical significance. This lecture is practically devoted to the study of an equation which is a generalization of the one given above. Professor Evans calls such equations integro-differential equations of the Bôcher type. By defining the adjoint of such equations, the author is able to prove an extension of Green's theorem for them. He now subjects the equation to an arbitrary real point transformation, and finds that the equation is transformed into a similar one; and, furthermore, that a certain differential quadratic form $T(dx, dy)$ is a covariant of the transformation. The directions defined by $T(dx, dy) = 0$ are called *characteristic directions*, and the solutions of the equation are called *characteristic curves*. The characteristic directions may be real and distinct, real and coincident, or imaginary. The integro-differential equation is said to be hyperbolic, parabolic, or elliptic, respectively, just as in the theory of linear partial differential equations. Normal forms of these types are then obtained, and a rather detailed study of the parabolic type is made. The lecture closes with some remarks on the usual types of integro-differential equations, as originally given by Volterra.

Lecture V, the last of the series, gives an account of the various generalizations of the theory of integral equations. The lecture begins with a statement of some of the more important properties of Stieltjes' integrals, and an application of these to the study of a class of equations involving Stieltjes' integrals. An existence theorem of such equations is given, but with the hypotheses as stated, an extension of the Fredholm theory is not possible.*

In generalizing the theory of the linear integral equation, one may

* It might be stated here that such a generalization has recently been given by F. Riesz in a very important paper in the ACTA MATHEMATICA (vol. 41 (1916), p. 71), which unfortunately appeared just too late to be included in these lectures. He uses there the concept of *completely continuous linear functionals*; it was this notion that made the extension possible.

proceed along at least two distinct lines. In the first place, one may, with E. H. Moore, generalize the variables, the classes of functions, and the linear operations, in such a way that the methods of Fredholm apply for the extended concepts. Or one may proceed, as Volterra has done in his *Theory of Permutable Functions,* to build a calculus of composition, i. e., of the operations which produce the iterated kernels. He is able by this means to solve an extensive class of non-linear integral and integro-differential equations. These extensions Professor Evans takes up in the order mentioned, and it should be remarked that an exceptionally clear exposition of Moore's theory is given, and a number of important contacts with the classical theory are pointed out. He notes, for example, that when the general range P is the one-dimensional continuum, and the class M is the totality of continuous functions over this range, Moore's linear operation J reduces to the classical Stieltjes' integral.

In the presentation of Volterra's theory, Professor Evans makes a considerable advance over the treatment found in Volterra's *Leçons sur les Fonctions de Lignes.* In the first place, he combines the theories of permutable functions of the first and second kinds into a single theory. In the second place, he introduces notations and concepts that simplify materially the proofs given by Volterra. This theory of permutable functions leads immediately to a very interesting extension of every analytic function. This extension carries with it corresponding extension of addition theorems, moduli of periodicity, etc. One such generalized function, called the *Volterra transcendental,* plays an important role in a certain class of integro-differential equations.*

It can be seen from this outline what a multitude of topics has been discussed in these lectures. After reading this book, one feels that the author is profoundly at home with every phase of the theory of functionals, to which he himself has made many important contributions. Although a serious effort has been made to give a clear presentation of these subjects, one cannot help feeling that it might have been wiser to have included fewer topics and to have developed each of them in a more leisurely fashion.

Of the misprints, one may mention those which have been noted already by Professor Evans in this BULLETIN (vol. 25, p. 461) and the following. In the theorem at the top of page 7, change $F'[\varphi(x) \mid \xi]$ to $F'[\varphi_0(x) \mid \xi]$. The left member of formula (24′) on page 16 should read $g'[c \mid ABM]$ instead of $g[c \mid ABM]$. On page 69, the right member of formula (33) should read $F[\varphi + \Delta\varphi] - F[\varphi]$.

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* See Schlesinger, *Zur Theorie der linearen Integro-Differentialgleichungen,* JAHRESBERICHT DER VEREINIGUNG, vol. 24 (1915), pp. 84–125.

The author remarks in his preface that the material has been so arranged that "the text in large type ... may be read by itself". Yet we find on page 34 (large type) references to equations (7) and (8), which are in small type. All these, however, are unimportant oversights, and the reviewer turns from this work with the feeling that there is in it a wealth of valuable information on practically every phase of the theory of functionals, with many suggestions for its future development. The bibliographical list that heads each chapter is a feature of not inconsiderable value.

I. A. BARNETT

JUNG ON ALGEBRAIC FUNCTIONS

Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen, by Heinrich W. E. Jung. Berlin, Walter de Gruyter, 1923. 246 pp.

The three great paths in the study of the algebraic functions of a complex variable—the geometric, the analytic, and the arithmetic—have as a common starting point a single algebraic equation, $f(x,y) = 0$. The traveler on one of the roads, once away from the point of departure, is often far out of hailing distance from those on the other paths; yet he is at times agreeably surprised to find he has reached the same point as they. At such times there will be a sign-post telling him and his fellow-climbers that they have reached the Riemann-Roch Theorem, it may be, or the Lückensatz of Weierstrass. Whatever the point to which the various paths converge, it is almost certain to be concerned with *genus*, or *deficiency*, if another language is used.

Multiplicity of dialects is, indeed, characteristic of the study in question. Not only has each path its own vocabulary, but the arithmetic path, with which we are chiefly concerned here, has no single valid language.* In Jung's book, for instance, we miss the mention of *Ring, Führer, Ideal, Modul, Integrabilitätsbereich, Polygon*, although most of the concepts named find a place. On the other hand, certain terms are borrowed from algebraic geometry, in particular, *canonical class* (corresponding to the canonical series) and *adjoint functions* (corresponding to adjoint curves).

* For a comparison of the content, and to some extent of the language, of the various theories, see Emmy Noether, *Die arithmetische Theorie der algebraischen Funktionen einer Veränderlichen, in ihrer Beziehung zu den übrigen Theorien und zu der Zahlkörpertheorie*, JAHRESBERICHT DER D. MATH.-VEREINIGUNG, vol. 28 (1919).

Although other paths to the study of algebraic functions have probably greater beauty and historical importance, the arithmetic approach can justly claim to be the most logical one. For the arithmetic theory alone concerns itself primarily with the functions, not with curves or integrals. And Jung's book does much to increase the esthetic claims of his route.

Hensel and Landsberg's scholarly treatise on the arithmetic theory* being the only predecessor of Jung's *Einleitung*, a comparison of the two is valuable. The newer book is much the shorter. Brevity is obtained by assuming somewhat greater mathematical knowledge at the outset, and by restricting the scope of the study. The reader is expected to be acquainted with much of the theory of functions of a complex variable, in particular, with facts on development of functions in power series and on the inversion of functions so developed, and with essential singularities. Riemann surfaces are, however, explained from the beginning. Knowledge is assumed of the Lagrange interpolation formula, a little of the theory of matrices, and both point and line (homogeneous) coordinates in the plane.

Chapter I deals with an algebraic function of x defined by a single equation and with its behavior (i. e., development in power-series) in the neighborhood of any point. An important concept introduced here is that of the *Einheit* (we may call it an *E*-function in English), which, at a given point, is finite but not zero.

Chapter II gives methods for computing the power-series for an algebraic function at any point. Puiseux's diagram is clearly and fully explained. We could wish, in both these chapters, for more attention to convergence proofs.

The next chapter is devoted to an admirable exposition of the Riemann surface. Particularly clear are the figures, the account of the identification of branches, and the summary.

The preliminary apparatus having thus been set up, Jung proceeds, in Chapter IV, to the algebraic body (xy); this consists of all rational functions of x and of y, the variables being connected by the algebraic equation $f(x, y) = 0$. The essential properties of the functions of the body are then developed: unique definition on the Riemann surface, equality of numbers of zeros and of poles, representation in power series, etc. The residue of a differential $\eta d\xi$ appears in this chapter (η and ξ are functions of the body), as well as a few theorems preparatory to the reappearance of the residue in the chapter on integrals.

Chapter V is entitled Birational Transformations. There is in the arithmetic treatment no thought of the (x, y) plane, no occasion, accord-

* *Theorie der algebraischen Funktionen einer Variabeln und ihre Anwendung auf algebraische Kurven und Abelsche Integrale.* Teubner, Leipzig, 1902.

ingly, for one-to-one transformations of that plane. The chapter centers about the theorem that, if ξ is a variable of the body (xy), there exists another variable η of (xy), such that the bodies (xy) and $(\xi\eta)$ are identical.

In Chapter VI we come to what is probably the most characteristic tool of the arithmetic theory of algebraic functions, the *divisor*. Strangely enough, one is never told *what* a divisor is. To every place (Stelle) $\mathfrak{p}$ of the algebraic body "is assigned" a prime factor with the same name $\mathfrak{p}$. The essence of a prime factor is defined as follows. If t is a function with a zero of lowest possible order at $\mathfrak{p}$, and if a function R is the product of t^λ and an E-function at $\mathfrak{p}$, then R is said to be *divisible* by $\mathfrak{p}^\lambda$. A prime factor is, one might say, the minimum amount of zeroness at a point. A divisor is a product of integral powers (positive, negative, or zero) of prime factors. A divisor is determined, for instance, by the zeros and poles of a function. All divisors are grouped in classes; such a class consists of products of a single divisor and of all divisors of functions. The latter constitute, of course, a class by themselves, the principal class; while differentials, $\gamma d\xi$, furnish another, the differential or canonical class. Linear independence of integral divisors is ingeniously defined, and the number of linearly independent integral divisors in a class is called its dimension. It is a misfortune, probably irremediable, that this number is one more than the dimension of the equivalent linear series of groups of points on a curve, — that two languages use the same word in senses just different enough to be misleading. In this chapter we notice an uncertainty as to what an α-fold branch point is. On page 71, line 1, a three fold branch point is one where four sheets join; yet on page 73, line 24, α sheets dance around an α-fold branch point. "Exactly divisible", a phrase introduced on page 79, ought to be defined. In this chapter the genus enters, defined by the number of sheets and by the sum of orders of branch points.

Chapter VII is devoted to matrices. It is hard to see why matrices should be called "systems" throughout this chapter, and developed from the beginning, when Jung will later (p. 154) not only talk unconcernedly of a matrix, but assume that the reader knows facts about it not mentioned in Chapter VII. The treatment is, at any rate, very satisfactory. The introduction of "multiplicative" and "additive" matrices for producing, on multiplication, desired changes in given matrices, is decidedly ingenious. Other important types are the "Einheiten", which we might now translate as "E-matrices". They are square matrices of rational functions of x, whose determinants behave as E-functions, either at all points, or at a single one. It seems unfortunate that matrices should be walled in indiscriminately by parentheses and braces. As parentheses are also used by Jung to indicate a class of divisors,

braces to indicate its dimension, the double bar before and after would certainly be better, if the printer could be persuaded.

The eighth chapter treats the problem of determining all multiples of a given divisor; in the language of other writers the totality of such multiples is an "ideal". The solution is reduced to the determination of a "base" for the functions divisible by the divisor — a group of functions such that, in linear combination with rational coefficients, they generate the body (xy), whereas with integral rational coefficients they generate the multiples of the divisor. The task of showing that such a base can always be found is a strenuous one, the chapter correspondingly long and difficult; but the work is done in masterly fashion. There are, at the end, two sections on complementary bases: pairs of bases, such that the two matrices of the functions and of their conjugates are complementary.

The final chapter of the purely arithmetic theory is devoted to the Riemann-Roch Theorem. As here given, it is concerned with the dimensions and orders of two "Ergänzungsklassen", i. e., classes such that a divisor of one and a divisor of the other have as product a divisor of the canonical class. The central position of this theorem arises from the information it furnishes concerning the existence of functions with given poles and zeros. Several consequences of the theorem for bodies of genera 0, 1, and 2 are deduced, and the chapter ends with the introduction of Weierstrass points.

Two chapters are now devoted to the geometry of the curve $f(x, y) = 0$. In one of them, non-homogeneous coordinates are used, and in the other the change to homogeneous coordinates is made. Many headings of sections, such as multiple points, form of real branches, number of intersections, Hessian curve, Plücker formulas, are those found in any book on projective geometry. The treatment, however, using the divisor on almost every page, gives to familiar facts unfamiliar clothing. Plücker's formulas, for example, are equations between orders of divisors.

Just as there is, in the chapters on geometry, an adequate transition to the geometric theory, so we find, in the last two chapters, an introduction to the integral theory. They contain, respectively, a treatment of the analysis situs of Riemann surfaces, and a short discussion of the three kinds of abelian integrals. The writing is clear and well-balanced; yet one feels Jung's realization that his chief contributions to the mathematical reading public had been made earlier in the book.

If I had mentioned the book's chief claim to distinction in any single chapter, I should have needed to repeat it in nearly all. About one-fourth of the space is devoted to examples, carefully chosen and clearly worked out at more or less length. The student who merely checks the results given will have a better knowledge of the real meaning of the theory than one who reads, however carefully, the

scholarly book of Hensel and Landsberg. Often there are gaps and suggestions enough to furnish the reader with material for real study; and every example worked out can, of course, suggest other problems to be solved ab initio.

Among the misprints and slips which I have noted, the correction of the following will be of importance to the reader.

PAGE	LINE	FORMULA	FOR	READ
9	30		des in ihm liegenden	eines in ihm liegenden
22	13		keine der anderen Strecken	alle anderen Strecken
49	5		$arg\ x$	$arg\ (x-a)$
72	5		wo t eine Einheit	wo $E(t)$ eine Einheit
80	8		Kap. VIII	Kap. IX
102		9	$g(x) \pm h(x)\, y$	$g(x) + h(x)\, y$
106	7		(22)	(18)
165	3rd row, 2nd matrix, middle of page,		0100	0010
173		5	$\frac{\partial f}{\partial x} = \frac{\mathfrak{d}_{3y}}{\mathfrak{l}^{-2}\mathfrak{m}^{m}}$	$\frac{\partial f}{\partial x} = \frac{\mathfrak{d}_{3y}}{\mathfrak{l}^{t-2}\mathfrak{m}^{m}}$
186	27		die p_k definierende	die p_1 definierende
191	26		Ebenenkoordinaten	Linienkoordinaten
195		30	$-a_3 da_1$	$-a_3 da_2$
196		38	$\mathfrak{d}$	—
197		39		$\mathfrak{d}$
211	24		$\lim \frac{\xi}{\eta} = -4$	$\lim \frac{\eta}{\xi} = -4$
235	19		$= \lambda$	$= \lambda + p$
238		15	ρ	p

Jung's book is adequately and clearly illustrated, fully and accurately indexed. His style is good. The prospective foreign reader will be particularly encouraged by the fact that the average sentence is but two lines long. It is to be hoped that this book will have many readers in America, in spite of the obstacles which our educators' continuing slight of the German language puts in the way of our next generation's progress in science. A moral may be learned from the fact that Germans have almost a monopoly of the work published on the arithmetic theory of algebraic functions.

All in all, we can say that Jung has chosen a task well worth doing, and has performed it excellently.

E. S. ALLEN

KLEIN'S COLLECTED PAPERS, VOLUME III

Felix Klein: *Gesammelte mathematische Abhandlungen. Vol.* III: *Elliptische Funktionen, insbesondere Modulfunktionen*; *Hyperelliptische und Abelsche Funktionen*; *Riemannsche Funktionentheorie und automorphe Funktionen.* Edited by R. Fricke, H. Vermeil, and E. Bessel-Hagen. Berlin, Julius Springer, 1923. IX + 774 + 36 pp.

The editors of the third volume are three in number; those of the second having continued, and been joined by the labors of a third collaborator. The plan of the preceding volumes has been carefully preserved. To give some idea of the care with which the work of the editors has been done, it may be mentioned that every formula has been rederived, and in cases of any discrepancies in notation or in result, these are fully explained in appropriate footnotes. The volumes are practically free of typographical errors.

The memoirs in the first part, mentioned under the first subtitle, are fourteen in number, being memoirs LXXXI to XCIV inclusive. Of these, the first eight were written during the period 1877–1880, while the author was in Munich; the next four, one from 1881, the other three in 1885, were written in Leipzig, and the last two, 1893, 1896, in Göttingen. This first part occupies 314 pages.

When Klein went to the technical school in Munich from Erlangen, in 1875, his main interest was in the use of elliptic functions in the solution of the quintic equation; in particular, what part the theory of the icosahedron could play in such solution. His courses in the technical school included a repeated cycle in the theory of numbers, elliptic functions, and algebraic equations. In the seminary he had such men as Pick, Gierster, Dyck, Fricke, Hurwitz, and for one semester, Bianchi. Not only did these men contribute material assistance at the time, but except the first and last, wrote their dissertations under Klein's direction, and have very considerably enriched the literature by their own contributions.

The first memoir is an extract of a letter, in Italian, to Professor Brioschi, showing how certain formulas used by Klein could be reduced to those previously used by Kronecker, Brioschi and Hermite. The second, on the transformation of elliptic functions and the solution of the quintic equation, contains the systematic development of the method, and the result, in abstract form, that the general solution of the quintic equation can be expressed in terms of elliptic functions. The author apologizes that his paper contains so much that was already known, but it was necessary to reproduce much of the known theory, to show

the relations between the various parts, and in particular the meaning for his method of procedure. The result is a systematic presentation which serves splendidly to introduce the reader to the whole Klein scheme of the theory of functions, in particular the conformal mapping, extended from Schwarz's paper on the hypergeometric series.

The next paper considers the reduction of the modular equation by appropriate transformations. In this paper it appears that transformations of elliptic functions of orders $n = 5, 7, 11$ offer particularly promising results. The details for $n = 5$ are included in the paper itself. Besides the modular equation of order 6 and its known quintic resolvent, we also have, as Galois resolvent of the latter, the icosahedron equation of order 60.

The next paper, transformations of order seven of elliptic functions, brings the case $n = 7$ to the same degree of completion. The Galois resolvent is now of order 168; its roots have the property of remaining invariant under the linear fractional substitutions of the ratio η of the periods of the elliptic function, the coefficients of which are congruent to the identity, modulo 7, and only these. The next problem is to determine the relation between this root and the absolute invariant. We know that the genus of this equation is 3, and that it cannot be hyperelliptic. Hence the equation can be reduced to represent a non-singular plane quartic curve which remains invariant under a birational group of order 168. Moreover, the p adjoints of order $n-3$ go over into the p adjoints of order $n'-3$ by every birational transformation which sends the given curve of order n into one of order n'; hence in our case, the operations of the group are linear. But from Jordan's list of finite groups of ternary linear substitutions it appeared that no such group existed. Either Klein's premises were wrong, or Jordan had made an error. Subsequent consideration established the existence of the group, which should therefore be added to Jordan's list.*

From the singular values of the crossratio, in the linear fractional substitutions of η, it appears that our curve can have but three sets of points which have fewer than 168 conjugate positions; these exceptions appear by sets of 2, or 3, or 7. The first are the sextactic points, the second the points of tangency of the bitangents, and the last are the points of inflexion. But there is only one set of 24 points, hence the residual intersection of each inflexional tangent with the curve must be another point of inflexion. By choosing three associated inflexional tangents for the sides of the triangle of reference, the

* C. Jordan, *Mémoire sur les équations différentielles linéaires à intégrale algébrique*, CRELLE'S JOURNAL, vol. 84 (1878), pp. 89–215. The error is on page 167, line 8 from below. The expression Ω should be divided by 3φ instead of 9φ.

equation of the curve is readily determined. The arrangements of the double tangents, the points of inflexion and sextactic points complete this part of the problem. The later sections of the paper are devoted to the construction of the Riemann surface, and of the curve itself, the above coordinate system being an equilateral triangle.

This paper is a gem of mathematical writing. Although it contains a wealth of new, and many unexpected results, the thought is so developed that each is derived directly and naturally, and each step suggests how the next should be taken. Instead of being lost in a maze of formulas — particularly likely to be the case in this branch of mathematics, — all the results are obtained with a minimum of manipulation, and by steps that leave nothing to be desired as to rigor or conclusiveness. The most striking feature is the ready and easy use of so many different branches of mathematics; here we find projective geometry, binary and ternary invariant theory, theory of groups, elliptic functions, conformal mapping, differential equations, algebraic curves — brought into service, each contributing its part in the general problem. This is everywhere characteristic of Klein's writings, but perhaps it reached its highest manifestation in this and the following papers. The case $n = 11$ was treated immediately, but the generalization to n any odd prime appeared several years later, after the author had been intensely occupied with more general questions.

Each of these memoirs is followed by a few pages of comments, largely concerned with subsequent studies by other authors. These are followed by two general reports on the theory of elliptic functions and modular functions, written near the end of the Leipzig professorship, then follows a short note on the composition of binary quadratic forms, and a longer outline of the autographed notes on the course on the theory of numbers given at Göttingen in 1895–96. It is shown that by proper interpretation of certain non-euclidean concepts of measurement the definite and the indefinite forms can be treated in the same way. Later the forms with negative discriminant are applied to elliptic functions.

The second part consists of three memoirs, two on hyperelliptic functions, and the third on abelian functions. The latter has two rather distinct parts, the first being general, and the second, which carries certain considerations much further, is restricted to $p = 3$. All these memoirs were written in Göttingen, 1886–1889. Beginning with the summer of 1887, Klein lectured two semesters on hyperelliptic functions, and the three following ones on abelian functions. During the last two years at Leipzig Klein had directed seven doctor's theses in this field, among them Fricke, and during the first few years at Göttingen as many more. These assisted materially in providing con-

firmations of details which Klein felt were correct, but which he had not taken the time or trouble to verify himself. Even then a great number of formulas were left undeveloped, and many numerical factors unprovided, which were supplied later by Burkhardt for the hyperelliptic case in two splendid papers.*

In the case of the abelian functions, the author mentions that he received the greatest assistance from the two students W. F. Osgood and H. S. White. The purpose of these papers is to bring the theory of hyperelliptic and of abelian functions into systematic and natural relation with the theory of invariants. In order to make this possible, the number of homogeneous variables could not be prescribed, and the domain of rationality had to be defined in each case.

The main result in the first paper is a systematic treatment of the theta functions, in particular for $p = 2$, and to determine the theta constants (for zero arguments) as modular functions. The associated hyperelliptic curves can be expressed in a definite canonical form, which suggests and makes possible a uniform treatment. No corresponding form exists for the non-hyperelliptic curves, and a general treatment is probably not possible. In the former, Klein developed the entire theory of the sigma functions; for abelian functions this was possible only for $p = 3$. The general form of the associated plane quartic curve is presupposed, and all the steps are taken in terms of invariants and covariants, rather than as applicable only to a particular system.

In the second part, a great deal of the paper is taken up with the systems of contact lines (double tangents), conics and cubics, and the necessary modifications when the curve has a double point. While this work is very interesting, and perhaps of considerable value in the study of the theta functions, and their geometric meanings, practically all these theorems on the quartic curve can be established directly from the study of the (1, 2) correspondence associated with the Geiser involutorial transformation. An interesting application is to hyperelliptic surfaces of genus 3, as there are 36 non-equivalent systems of canonical sections that make the surface simply connected. The author mentions that he was indebted to H. D. Thompson for appropriate graphical schemes to show the relations between these various systems.

The third division of the volume occupies about three hundred pages.

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* *Beiträge zur Theorie der hyperelliptischen Sigmafunktionen*, MATHEMATISCHE ANNALEN, vol. 32 (1888), pp. 381–442, and *Grundzüge einer allgemeinen Systematik der hyperelliptischen Funktionen erster Ordnung. Nach Vorlesungen von F. Klein*, MATHEMATISCHE ANNALEN, vol. 35 (1889), pp. 198–296. The later publications in volumes 36, 38, 41 of the same periodical represent further developments of Burkhardt's own ideas.

Chronologically, it precedes the one just mentioned, as practically all of it was written while Klein was at Leipzig. We are told in the introduction that Klein regards this as the most productive period of his activity, as to quantity and as to importance. The beautiful results obtained in the study of the transformation of elliptic functions led the author to a more ambitious program of an exhaustive study of the groups of linear fractional substitutions of one variable, and then of the functions which remained invariant under such substitutions. The method was to be that of Riemann, with a minimum of analysis and a systematic application of physical ideas, strengthened by a highly developed intuition, both of spatial relations and of physical phenomena. Although himself a pupil of Plücker, and later in Göttingen greatly assisted and directed by Clebsch, Klein emphasized throughout his career that he was more influenced by Riemann than by any other source.

At Leipzig he at once began a course on the geometric theory of functions, to extend over several semesters. The lecture notes of the first semester were not published, but several copies were in the hands of students, and the dissertations of Gierster, Hurwitz, Staude, Lange and Weichold were begun. The substance of the second semester's lectures was put into book form and published under the title *Über Riemann's Theorie der algebraischen Funktionen und ihrer Integrale* by Teubner in 1882. It is given in full in this third division. There are no additional comments on the text, but at its end appears a note which calls attention to the similarity of methods here developed and those employed by F. Schottky. In reponse to an inquiry of Klein as to the source of his ideas, Schottky replied that it was the study of the circulation of an incompressible fluid—that is, Riemann's idea. Attention is also called to the fact that in Weyl's book (*Die Idee der Riemannschen Fläche*) most of the concepts used by Klein have been confirmed by the rigid requirements of modern analysis.

An algebraic function was defined by means of its Riemann surface, and the latter was interpreted as any closed surface in space. The rational functions associated with the rotations which leave the regular bodies invariant, trigonometric, and elliptic functions were all well known examples of functions belonging to the group to be studied. The modular functions furnished a large advance, as they belonged to infinite groups in which the coefficients are integers with determinant unity.

While these ideas were rapidly maturing, but before the broad lines of the general theory were published, there appeared, in rapid succession, a number of brilliant notes in the Comptes Rendus, 1881, by H. Poincaré, on certain phases of the same subject. These notes prompted Klein to write to Poincaré, primarily to protest against the name (*Sur les fonctions fuchsiennes*) and to call to his attention a considerable number

of earlier memoirs in which he (Klein) had developed the same ideas. The name *Fuchsian* had been applied by Poincaré to those functions which have a fundamental circle; the matter was not made more pleasant when he proposed and later used the adjective Kleinéens, to describe the functions which do not have a fundamental circle. Now follow twenty-five letters, all written during the years 1881, 1882. Klein invited Poincaré to prepare a synoptic paper on his work, and proposed that he would also prepare a similar paper. They both appear in volume 19 of the MATHEMATISCHE ANNALEN, that of Klein appearing in the present volume, covering eight pages. It was followed, six months later, by the principal memoir Klein wrote on this subject, *Neue Beiträge zur Riemann'schen Funktionentheorie*, which appeared in the MATHEMATISCHE ANNALEN, vol. 21 (1882-83), and occupies 80 pages. Of these, the last six are occupied with modifications and refinements of the existence theorem that on every given Riemann surface there exists one and but one principal function. The proof is not completed; indeed in various parts of the paper the argument is somewhat sketchy. It was at this time that Klein's health broke down. He suspended work for some time, and when he could resume, it was only on a restricted scale, for several years. Apart from a few notes, this is the last memoir on automorphic functions written over his own name, but in the preface to the treatise on *Automorphe Funktionen*, Fricke explained that he was directed by Klein through most of the great undertaking. Near the end of the present volume appears a critical analysis of the various proofs of this theorem, prepared by Klein and Bessel-Hagen, and a similar criticism of the origin and development of the *Automorphe Funktionen*.

In the appendix of 36 pages a number of lists are collected, as follows: University courses and seminar themes, 1871-1923, complete list of the 48 doctor dissertations written under his direction, together with the time and place of publication of each, list of his assistants, and a list of his 151 memoirs, 5 commemorative address, 3 reviews, 10 books, 10 autographed volumes, 33 reports on the organization of the teaching of mathematics, and his work as editor of the MATHEMATISCHE ANNALEN, the ENCYKLOPÄDIE, the International Commission, and the works of Plücker, of Möbius, and of Gauss.

These three magnificient volumes, clearly printed on high quality paper, filled with instructive personal notes, generously stocked with references to the work of others, even when cherished ideas of the author were first published by others, all skillfully compiled by patient and sympathetic hands, almost completely free from typographical errors, furnish an almost invaluable commentary on the development of mathematics during the last half century.

VIRGIL SNYDER

BIANCHI'S DIFFERENTIAL GEOMETRY

Lezioni di Geometria Differenziale. Terza edizione interamente refatta in due volumi. Vol. I. By L. Bianchi. Pisa, Spoerri, 1921. IV + 806 pp.*

The first and second editions of this classic, whether in Italian or German, are so well known to all students of differential geometry, or should be, that an extended review of this edition would be an anachronism. When one turns over the pages he is reminded of the earlier editions, but when he compares it with the latter he observes much that is new in the development of the topics formerly treated. And so he decides that hereafter, when he wants to see what Bianchi says about a certain subject (and a student of differential geometry cannot afford not to do so), he will use the new edition.

The first ten chapters of the second editions, both Italian and German, are retained in this volume, but they have been developed to such an extent that they occupy half again as many pages.

The problem of reducing the linear element of a surface to the form $ds^2 = du^2 + 2\cos\omega\, du\, dv + dv^2$ is called the *problem of Tschebychef* by Bianchi, and the parametric curves the *lines of Tschebychef*. This problem was discussed briefly on page 401 of Volume 2 of the old edition, but the new treatment is more extensive. In fact, it is shown that such lines exist on any surface, the degree of arbitrariness being that of two arbitrary functions of one variable each (p. 159). More recently Bianchi† has proved that the tangents to either family of these lines, say $v = \text{const.}$, at points of any curve of the other family are parallel in the sense of Levi-Civita with respect to the latter curve; moreover, this is a characteristic property of the lines of Tschebychef.

In Chapter V it is shown that the Christoffel symbol of the second kind formed with respect to the second fundamental form is the arithmetic mean of the corresponding symbols formed with respect to the linear element of the surface and of its spherical representation. Also there are certain results concerning spaces of constant Riemannian curvature of n dimensions (p. 259).

In 1869 Dini solved the problem of geodesic representation of two surfaces upon one another. In 1896 Levi-Civita extended the problem to spaces of any order. The results of Dini and their application to

* The first German edition was reviewed by J. K. Whittemore in this BULLETIN, vol. 7 (1901), pp. 431–442, and the second German edition by L. P. Eisenhart in vol. 18 (1912), pp. 411–418.

† BOLLETTINO UNIONE MATEMATICA ITALIANA, vol. I (1922), pp. 11–16.

the determination of pairs of quadrics in this relation have been added to Chapter VI.

When two surfaces are applicable, the curves on one corresponding to asymptotic lines on the other have been called *virtual asymptotic* by Bianchi. The differential equations determining such lines on any surface were derived by Darboux, who showed that the problems of finding these lines on a surface S and the surfaces applicable to S are equivalent. This theory is set forth at the close of Chapter VII; it appears at the same place in the second German edition but not in the Italian one.

In his beautiful theory of surfaces applicable to a quadric Bianchi made use of the theorem of Chieffi: If at the point of any asymptotic curve C upon a surface S applicable to a ruled surface R one draws tangents to the geodesics g of S which are deforms of the generators of R, the ruled surface R_1, formed by these tangents is applicable to S with g remaining rigid. Bianchi adds in § 153 an analytic proof of this theorem to the geometric proof given on page 4 of volume 3 and in § 126 of the second German edition.

Twelve years ago Sannia published several papers in the RENDICONTI DI PALERMO dealing with the intrinsic definition of rectilinear congruences by means of two quadratic forms. Bianchi devotes §§ 187 and 188 to this theory and to an application to congruences whose mean evolute is a point.

The general theory of relativity has aroused a general interest in the differential geometry of Riemann spaces of any order. The last half of the first volume of the second Italian edition contains an excellent exposition of this theory. This is omitted from the volume under discussion with the exception of the chapter dealing with "pseudospherical geometry and its non-euclidean interpretation", which appears as Chapter XIV. However, the last half of the second volume will be devoted to general Riemann geometry, and we can expect that it will contain some of the recent additions to this live subject.

Chapter XI is devoted to surfaces with plane or spherical lines of curvature, which appeared formerly as Chapter XXI with some omissions and additions. Chapters XXII and XXIII of the second Italian edition, or XIV and XV of the German edition, dealing with minimal surfaces, appear with little change as Chapters XII and XIII.

Bianchi began his career as a student of pseudospherical surfaces, and the theory of these surfaces and their transformations owes to him its present state of perfection. This subject has always seemed particularly fascinating to him. And now here in the last two chapters of this volume we have his exposition of surfaces of constant curvature based upon the knowledge and experience of a lifetime. They are extremely well done.

Anyone who is familiar with the method of the moving trihedral,

introduced by Darboux, and who has appreciated its power in dealing with geometrical problems, must wonder why Bianchi has never made much use of it. He introduced the elements of it in his treatment of Weingarten's method for the study of applicable surfaces (vol. II, p. 176), but seems not to have used it elsewhere. It appears as a note at the close of this volume.

Bianchi is a past-master in the art of writing treatises, not only in the field most closely associated with his name, but in many other fields. And this book reveals him at his best. The reader has no grounds for criticizing the book as to clarity of statement, but the student who wishes to follow up a particular subject may regret the lack of references to many of the sources from which the cream has been taken.

L. P. EISENHART

THE IMUK REPORTS AND LOREY ON INSTRUCTION IN GERMAN UNIVERSITIES

Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19. Jahrhunderts. Von W. Lorey. (Abhandlungen über den Mathematischen Unterricht in Deutschland veranlaßt durch die Internationale Mathematische Unterrichtskommission.* Band III, Heft 9). Leipzig, Teubner, 1916. 12 + 431 pages and 4 plates.

During the deliberations of the fourth International Congress of Mathematicians at Rome in 1908, steps were taken to organize an International Commission on the Teaching of Mathematics, the members of which were to prepare or procure reports on the methods and materials of mathematical instruction in different countries. Most of these reports were ready when the fifth International Mathematical Congress convened at the University of Cambridge in 1912, but several more have appeared since then. At this writing, 18 countries have published 294 reports containing over 13,500 pages. Germany has issued 53 reports with a total of 5571 pages; about one-fourth of this space is required by the United States for its 18 reports, and about one-sixth of the same space by each of the following countries: Austria for 13 reports; Great Britain for 39; Switzerland for 13; and Japan for 16 reports in two volumes. The reports of France cover nearly 700 pages. Of more modest dimensions are, in order of size, the reports from Belgium, Russia (including Finland),

* A new word *IMUK* has been coined and is used in German writings as an abbreviation for these last three words.

Hungary, Italy, Sweden, Spain, Holland, Denmark, Australia, Argentine, and Roumania (one report of 16 pages).

The German reports have been published under the general heading: Schriften des deutschen Unterausschusses der IMUK. They include Abhandlungen, edited by F. Klein, and Berichte und Mitteilungen, edited by W. Lietzmann, of which the first series appeared in 12 numbers during the years 1909–1916 and extends to 356 pages. Two numbers of the second series are the reports on mathematical instruction in Denmark (1915; 60 pp.) and England (1915, 211 pp.).* The third and last volume (1917, 115 pp.) of this second series contains a complete index to both the Berichte und Mitteilungen and the Abhandlungen.

The Abhandlungen (38) by 33 different authors (1909–1916), and containing over 4800 pages, are now complete and occupy five (in reality nine)† volumes. The general titles of these volumes are as follows: 1. Die höheren Schulen in Norddeutschland; 2. Die höheren Schulen in Süd- und Mitteldeutschland; 3. Einzelfragen des höheren mathematischen Unterrichts; 4. Die Mathematik an den technischen Schulen; 5. Der mathematische Elementarunterricht und die Mathematik an den Lehrerbildungsanstalten. The imposing Abhandlung under review is the last number of the third volume and it was the last one of all the Abhandlungen to be published‡ (preface dated "6. Juli 1916"). It is more than twice as large as any of the others.

For the American mathematician this volume will probably be by far the most interesting of the Abhandlungen. As in nearly all the other reports, the historical side is heavily emphasized; but here movements and developments are associated with institutions and men whose names are more or less familiar. Moreover the biographical outlines (some quite complete) provided in the case of every mathematician mentioned, are occasionally entertainingly supplemented by extracts from published sketches,§ and reminiscences especially prepared for this volume. Only two days before his death Heinrich Weber contributed an account of his student days.

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* This volume is decidedly useful for filling up many lacunae of the reports from Great Britain which are confined almost wholly to an account of conditions in England.

† Each of three Bände is divided into two Teile, and one of the Teile has two Abteilungen.

‡ A number of the countries have made no report on mathematical instruction in their universities, except in connection with account of the training of secondary school teachers.

§ The bibliographical references are numerous and valuable; much material rarely met with has been brought to light and placed conveniently for reference.

Interesting facts are given concerning eminent mathematicians, other than German, who influenced German mathematics; for example, Abel, Euler, Lie, Lagrange, and Steiner. Regarding such a man as Gauss, the formost German mathematician of the nineteenth century, there is very little, since the volume aims to give an account of mathematical instruction in the universities, and not a history of the development of mathematics as a science. Gauss was director of the observatory and professor of astronomy, not mathematics, at the University of Göttingen, from 1807 to the time of his death in 1855, and he "hat immer nur vor einem kleineren Kreise von Zuhörern und nie mit großer Freude gelesen."

The first chapter of 22 pages contains general information about the 22 German universities. Six of these were founded in the fifteenth century, the one at Leipzig, established in 1409, being the oldest. Four universities were founded in the nineteenth century, while that at Frankfort-on-Main was opened in the autumn of 1914. Just half of the universities are in Prussia.

There are sections on the faculties, the teaching force, and the students. Of the seven classes of teachers comprised in the 3724 professors and instructors for the wintersemester of 1914–15, about 3400 were ordinary and extraordinary professors and privatdozenten. The first and third of these groups were almost numerically equal and constituted about four-fifths of the whole. In mathematics there were 49 ordinary and 19 extraordinary professors and 22 privatdozenten. Of these 90 teachers Göttingen had 9, Munich 7, Strasbourg 6, and so on down to Rostock with 2.

The section on students contains two charts; one indicates the number of candidates for the Prussian Staatsexamen, with mathematics and natural science as major, between 1839 and 1913. From about 20 in 1839 there was a fairly constant increase to nearly 160 in 1882; this number dropped to less than 25 by 1893, but increased almost continuously to over 325 by 1913. The second chart shows the number of students of mathematics and natural science at the University of Göttingen from 1868 to the summer of 1913. The numbers increased from about 60 in 1870 to almost 240 in 1881, dropping to less than 80 in 1892 but increasing to over 750 by 1913. The intimate relation between the number of university students and the number of those taking the Staatsexamen is noticeable.

The rest of Lorey's work is divided into five chapters, 2–6. The second (26 pages) deals with the state of mathematical instruction in the universities at the beginning of the nineteenth century, and with vain trials of methodically arranged new organization in the years 1818–1848. The third chapter (pages 59–111) is entitled: "The development of mathematical instruction in the universities up to the

founding of the German Empire." The fourth chapter (pages 111-141) discusses "Special achievements in the period before 1870 with glances at the present time." The fifth chapter (pages 142–211) deals with "The period from 1870 to 1890. From the founding of the Empire to the beginning of the school reform movement." The sixth chapter (pages 211–402) is entitled: "Recent times (from 1890–1914)." Pages 403–8 are filled with "Corrections and additions" and pages 409–28 with a "Personen-Verzeichnis".

Although lectures in mathematics were given in German universities in the latter part of the eighteenth century, they were not universally regarded as feasible even by those who flourished in the early nineteenth century. A notable case is that of Mollweide (a friend of Gauss who characterized him as "ein gründlicher Mathematiker und trefflicher Mensch"), who did not think that higher mathematics could be properly dealt with in lectures "schon weil es dabei zuviel Schreibens an der Tafel gäbe!"

In the early part of the nineteenth century the universities at Königsberg (founded in 1544), Berlin (1810), and Göttingen (1737) were the three chief centers of mathematical inspiration. The first named university, especially, under the guidance of such notables as K. G. J. Jacobi, Bessel, and Franz Neumann, led to a well defined Königsberg school whose influence throughout Germany was great. For the 17 years 1826–42 Jacobi was constantly giving in his lectures results developed by prolific research. With Neumann he founded in 1834 a mathemathics-physics seminar, concerning which full and interesting details are given in the section on seminars in Chapter 4. About 1843 Jacobi moved to Berlin where he lived as a royal pensioner until his death in 1851. Jacobi's first pupil Richelot, a native of Königsberg, became a professor in the university and expounded the researches of his teacher and of Steiner. L. A. Sohncke and Otto Hesse were also natives of Königsberg. While the former served only two years as dozent in the university, the latter taught there as dozent and professor for 15 years. The influence of the Königsberg school was later extended when Hesse, Gustav Kirchhoff, H. von Helmholtz, and Bunsen were teaching together at Heidelberg University which "in ihrer höchsten Blüte stand". It was at this time that Heinrich Weber was a student, and his interesting reminiscences of the period have been preserved by Lorey.

Weber's early training doubtless influenced him in accepting the call to the University of Königsberg where he was professor for the ten years 1873–83.

In Berlin by 1835, Crelle (1780–1855), Lejeune Dirichlet (1805–1859), Dirksen (1792–1850), a pupil of Gauss, and Steiner (1796–1863) were prominent mathematical members of the Academy of Sciences. A little later Jacobi was also there. It was under Dirksen at the University

in 1825 that Jacobi, when only twenty years of age, passed a brilliant doctor's examination. Although Steiner had studied no mathematics and could scarcely even write at the age of 16 years, he rose rapidly into prominence by his publications during the years 1825-34 that he was teacher in the Gewerbeschule at Berlin. In 1832 on the recommendation of Bessel and Jacobi the philosophical faculty of Königsberg university conferred the honorary degree of doctor on Steiner. He was elected a member of the Berlin Academy in 1833, and became professor at the University of Berlin in 1834. He held this position until his death.

Of the three leaders (Dirichlet, Steiner, and Jacobi) of the first mathematical school at Berlin, Steiner undoubtedly exercised the greatest influence on his students and on geometrical instruction in schools. A great mass of ideas and propositions discovered or employed by Steiner penetrated into the teaching quite quickly, such as the theory of harmonic points and pencils, of the potency of a point, of pole and polar, the properties of the complete quadrilateral, and Pascal's Theorem.

Dirichlet was born in Aix-la-Chapelle. When 16 years of age, after finishing his secondary school education at Cologne, he went to Paris, where he studied mathematics and taught for six years. Through Fourier ne met Alexander von Humboldt, whose influence with the Ministry* resulted in his appointment as dozent in the University of Breslau in 1827. In 1829 he was called to the University of Berlin where he finally became ordinary professor in 1839. He remained there until about four years before his death, when he became Gauss's successor at Göttingen.

Crelle was not a professor in the university but a "Geheimer Baurat" who was, for example, responsible for the plans of the railway between Berlin and Potsdam. His numerous mathematical publications were not of the first rank, and his significance for mathematics lay in his energetic organizing activity. With Gergonne's ANNALES DE MATHÉMATIQUES PURES ET APPLIQUÉES as a model, he founded the JOURNÁL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK in 1826. Success in the undertaking was partly assured through an arrangement which he made with the Kultusministerium (ministry of ecclesiastical affairs and public instruction) whereby he was allowed to add to the title of the JOURNAL the words to be found even in recent volumes: "Mit thätiger Beförderung† hoher Königlich Preussischer Behörden." This Beförderung consisted on the one hand in issuing strong official recom-

* Although Alexander von Humboldt (1709-1859) was not a mathematician his name comes up several times in the Abhandlung as that of one very especially interested in the subject and wielding his powerful influence for the promotion of its study.

† For "thätiger Beförderung" Lorey has "Unterstützung".

mendation of the JOURNAL, not only to universities and higher schools, but also, for example, to government boards and, through Prussian ambassadors, in foreign countries. On the other hand the Beförderung involved the purchase of a number of copies of the JOURNAL which were distributed to various schools—a custom prevailing until the great war. In this way, Weierstrass, for example, while a gymnasium pupil, received inspiration by discovering an uncut copy of the JOURNAL "mit den schönen Abhandlungen von Steiner, von denen auch ein Primaner etwas verstehen konnte."

But in connection with the JOURNAL, the important thing was that Crelle's faith in young collaborators* was not misplaced. "Darin liegt gerade das unvergängliche Verdienst des damals schon bald fünfzigjährigen Crelle, daß er die jungen mathematischen Forscher, die in den zwanziger Jahren des vorigen Jahrhunderts in Deutschland auf einmal hervortraten, an sich heranzog und ihnen in seinem JOURNAL eine leichte Publikationsmöglichkeit verschaffte."

These indications as to the contents of the volume will suffice for suggesting its general character. It is an invaluable work of reference on the subject of which it treats.

R. C. ARCHIBALD

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* The first volume contained seven memoirs by Abel, five by Steiner, and one by Jacobi.

In spite of the title of the JOURNAL very few articles on applied mathematics were ever published in it. In 1829 Crelle founded another periodical JOURNAL FÜR BAUKUNST, which existed until 1851.

SHORTER NOTICES

Tractatus Logico-Philosophicus. By Ludwig Wittgenstein. With an introduction by Bertrand Russell. New York, Harcourt, Brace and Company, 1922. 189 pp.

The final number of Ostwald's ANNALEN DER NATURPHILOSOPHIE (1921) contains an article by Mr. Wittgenstein, a former pupil of Mr. Bertrand Russell, dealing with the nature of logic and with its relations to mathematics, philosophy, and natural science in a manner so original and profound as to make its publication an important event. The book in hand presents that essay in the original German, along with an English rendering of it under the editorship of Mr. C. K. Ogden of Magdalen College, Cambridge. The original and the translation are printed side by side, facing each other, and that is well, for the original contains many sentences that are not sufficiently clear to admit of quite confident translation.

Of the book's 189 pages 23 are occupied by Mr. Russell's introduction, the remainder being equally divided between the German version and the English one. So it is seen that the work proper is physically very small—only 83 pages. But it is far from small scientifically. Not only does it present "a theory of logic which," in Mr. Russell's opinion, "is not at any point obviously wrong" (notwithstanding it rejects as unsound some of the tenets hitherto held by the English logician) but it contrives to deal in a fundamental way with other great matters.

How can so small a work be so big? What is the art involved? The answer is found in a variety of considerations.

One of them is that Mr. Wittgenstein's thinking is confined to fundamentals. His book is addressed to none but the most seasoned of thinkers. The author will be content, he tells us, if only one person reads his book with understanding and pleasure.

Again, there is no index, no table of contents, no division into chapters, no bibliography, no specific acknowledgement of indebtedness to others save that of having been stimulated by "the great works of Frege and the writings of my friend, Bertrand Russell"; there is hardly any comparative criticism, setting the author's thought in relation with the thought of others, whether past or contemporaneous; and no clear indication of such parts of his work as he may deem to be new or original, "because," says he, "it is indifferent to me whether what I have thought has already been thought before me by another."

But the chief secret of his being able to deal effectively with so many great matters in so brief a space, is to be found in the temperamental quality of his style. Wittgenstein is a mystic—a logical

mystic—and like the great ones of that kind (Spinoza, for example, or Blaise Pascal), he is at once a slave of the propensity for condensation and master of the art. One may say of his style what Porphyry said of the style of Plotinus: "Dense with thought, more lavish of ideas than of words."

Wittgenstein's style is not admirable. His book is not an exposition; it is rather a conglomeration of insights, often profound, intimately related, wide-ranging, fit material for a magnificent structure, but they are not so ordered and presented as to constitute a luminous whole. In order to understand the book it is necessary to read it again and again, forward and backward, up and down, in and out. Even then, despite Mr. Russell's somewhat helpful introduction, some passages remain ambiguous, indeterminate, obscure; not because the subject is difficult, which it is, but because the author has not taken sufficient pains to be clear. Mr. Russell tells us that Mr. Wittgenstein's theory of logic "is not at any point obviously wrong." But upon the score of obviousness, he might have said with equal justness that the theory is not at any point obviously right. Mr. Wittgenstein deserves to be thanked for producing a book that every mathematical or philosophical logician must read, and to be at the same time reprimanded for allowing his lust for mystic condensation so to obscure his thought as to burden and sometimes to irritate the reader. Such a reprimand is not unjust, for it is of the very essence of the author's teaching that "everything that can be thought at all can be thought clearly" and "everything that can be said can be said clearly."

The central aim of the book is to answer a very important and very difficult question that has seldom been asked: What must be the essential nature of a logically perfect language? To present Wittgenstein's answer fully and clearly would require, as already said, more space than he himself has devoted to it. Yet even a brief review may give a few hints, serving perhaps to orient and stimulate the reader.

Consider the following random propositions: (1) the canary bird in the corner of this room has black eyes and a golden throat; (2) the specific gravity of mercury exceeds that of gold; (3) the velocity of light is greater than any other velocity. The propositions (whether true or false) are "symbols" representing "possible" facts. The facts, which are not propositions, "exist" or do not "exist" according as the propositions are true or false. If a fact, like that symbolized by proposition (1), is composed of two or more facts, it and the corresponding proposition are "molecular;" otherwise, "atomic." Even an atomic fact contains parts, for the fact is a relation among things (objects) and these are its parts. So, too, an atomic proposition has parts, for it is a relation among names (symbols for objects) and these names are its parts. Now, in a logically perfect language, each proposition has

a unique and definite meaning, but the meaning of a proposition is determined by the meanings of its parts (the names in it). A name denoting a "complex" object can have such a meaning only when the object has been completely analyzed; but complete analysis is possible, theoretically possible, only upon the assumption that a complex object is composed of "simples." So it is seen that the concept of a logically perfect language involves the hypothesis that every complex object is composed of simples and involves the requirement that each of the simples in the world shall have one and but one name, no name (of a simple) shall denote more than one simple (and of course no complex). Between all other names and the complexes in the world there must be a unique and reciprocal correspondence.

Language is composed of symbols: names symbolize objects, simple or complex; propositions (true or false) symbolize possible facts (existent or non-existent). The role of propositions is that of depiction (Abbildung), representation by means of pictures. How does a proposition symbolize—picture, depict—the fact asserted by it? In the philosophy of logic the importance of that question is fundamental, and Wittgenstein's answer is one of the gravest theses in his book. His answer is, in very brief, substantially as follows: A fact, being a relation among objects, has a certain structure, or form; a proposition (asserting the fact) also has a certain structure, or form, for it, too, is a relation—a relation among the names occurring in it and symbolizing the objects in the facts. Now, says Wittgenstein, the structure or form of the fact and the structure or form of the proposition (asserting the fact) are identical, and that is why the proposition is a picture of the fact. Moreover—and here we touch the nerve of the author's mysticism—a proposition cannot "express" its own structure or form, but can only "exhibit" it: the structure cannot be said, it can only be shown. The inexpressible is the mystical.

Wittgenstein's theory of logical inference is profound and beautiful. Space is lacking to set it forth here, for to do so would require a careful preliminary explanation of certain technical terms. It must be said, however, that the theory in question discriminates sharply between propositions of logic (including mathematics) and all other propositions. The former are true unconditionally, and are known to be true by inspecting them. All other true propositions are only conditionally true, and their truth cannot be recognized by inspecting them. Propositions of logic (including mathematics) are silent about the empirical realities of life and the world. But they are absolutely essential in the process of inferring from propositions that do relate to such realities to other propositions relating to them.

C. J. Keyser

Was ist Mathematik? Unterhaltungen während einer Seereise. By Lothar Heffter. Freiburg i. Br., Theodor Fischer, 1922. 160 pp.

Les Mathématiques. By Pierre Boutroux. Paris, Albin Michel, 1922. 183 pp.

Problems of Modern Science. Edited by Arthur Dendy. *Mathematics.* By J. W. Nicholson. New York, Henry Holt and Company, 1922. 237 pp.

Judging by the number of books that are appearing which devote themselves to the popular exposition of various branches of science, it would appear that the general public is becoming increasingly interested in the nature of scientific progress. Or is it only that publishers are beginning to realize the existence of this interest and are gradually becoming bold enough to meet the resulting demand? Of the books listed above the first two endeavor to give a popular account of mathematics as a whole; they are attempts to answer for the layman the question he so often asks: "What is mathematics all about, anyway? I know a little something about elementary algebra and plane geometry. But you people (meaning the mathematicians) talk about 'mathematical research', you use a lot of technical terms and a mass of mysterious looking symbols that nobody else can understand. Can't you tell something about it in terms that are intelligible to us?" The two books referred to offer replies to these questions in very different ways. The first is by far the less formal and is concerned primarily with ideas; the second is more formal and is primarily concerned with processes. The former can be more easily read, the latter is more adapted to study. Both cover a wide field.

The little book by Heffter is in the form of a dialog between a mathematician and a merchant, is delightfully written, and admirably accomplishes its purpose within the limits which the author has imposed on himself. "In the beginning was number" is the title of the third chapter (the first two are introductory) and here the author speaks genially of the concept of whole number (positive and negative), calls attention to the fact that there exist an infinitude of such numbers although each one is finite, illustrates the commandment "Thou shalt not divide by zero," discusses prime and composite numbers, congruence and quadratic residues and diophantine equations, not neglecting Fermat's last theorem and the Wolfskehl prize. The next chapter deals with the successive extensions of the concept of number up to and including the introduction of complex numbers, not omitting the Dedekind cut, the concept of transcendental numbers and the squaring of the circle, the fundamental theorem of algebra and the distinction between the existence of a root of an algebraic equation and its algebraic solvability. In Chapter V, the story of Achilles and the tortoise serves to introduce the notion of infinite series and products and continued fractions.

Chapter VI on "Connected variable numbers" introduces the function concept and the graph (including periodic functions and the depiction of one plane on another) and Chapter VII under the suggestive title of "Snapshot and moving picture" brings the concepts of derivative and integral and the fundamental theorem of the integral calculus, going so far as to introduce the notion of curvature and of differential and integral equations. The next three chapters are devoted to geometry, mainly projective (parallel and central projection, photograph, double ratio, points at infinity, duality, projective transformations and the group concept); but the fundamental notions of analytic geometry, imaginary elements, theory of surfaces (surfaces of constant curvature, non-euclidean geometry, minimal surfaces, lines of curvature and geodesics) are also discussed. In Chapter XI the fairy tale of the king and his five sons serves to introduce a problem in analysis situs and we are give a glimpse of the four-color problem, one-sided surfaces, and of concepts of genus and connectivity. The next chapter treats briefly of the classical mechanics and the last of the theory of relativity. It will be surprising to many that such a wide domain of advanced mathematical doctrine should be capable of informal and yet intelligible exposition to the layman, and yet it has been done and done extremely well. It would be highly desirable if this little book could be made available in an English translation; but would any American publisher be bold enough to publish it?

Boutroux's little book must be treated more briefly, not because it is less meritorious but because it will have a narrower appeal. As has already been suggested, it is more abstract in its treatment and is less easily digested by the lay reader. The author was himself well aware of this characteristic of his work and was apparently quite deliberate in bringing it about. In a ten page preface he discusses at length the difficulties in the way giving a popular exposition of mathematics and reaches the conclusion that in order to give a true picture of mathematics as it is the treatment must necessarily be rather severe. "It is desirable," he says, "that a book treating of mathematics be a bit abstract. If it demands more thought, the impression which it leaves will doubtless be more in conformity with truth." The character of the discussion is foreshadowed by this attitude on the part of the author. The chapter headings are as follows: I, Properties of Numbers; II, Properties of Figures; III, Mathematical Demonstration; IV, Algebraic Computation; V, Analytic Geometry; Functions and Derivatives; VI, Differential Equations; VII, The Theory of Functions. It will be seen that the emphasis is almost exclusively on analysis; there is no treatment of projective geometry or of analysis situs. The book would seem to be admirably adapted to the interested layman, who having read Heffter's little book, desires to go a bit further.

Professor Nicholson's article of thirty-seven pages in the last of the books under review has a different purpose than the other two, although it too is addressed to the layman. It forms one of a series of lectures delivered at King's College of the University of London on problems of modern science and attempts to present to a general audience an account of present activities in mathematical research with special emphasis on those which are related to other sciences. Any one faced with such a task has a perfect right to make his own selection of topics to be presented, and a reviewer has little right to criticise such selection. The present reviewer has no disposition to do so, beyond observing that other men would doubtless have made other selections. As is natural, the problems in vogue in England receive the greater part of the lecturer's attention. We thus hear of the theory of partitions, and of Waring's problem, and of related topics in the theory of numbers. A strange error has crept into the account of Fermat's theorem, when the founding of the Wolfskehl prize is located at Vienna. Dubious also is the statement (p. 19) that Fermat's theorem is one "towards the solution of which an extensive mathematical training is of little help." The last sixteen pages are devoted to some mathematical considerations connected with the theory of relativity and Planck's quantum theory. The lecture is readable throughout. It may be doubted, however, if a general educated audience would carry away more than a very vague notion of what it was all about, especially concerning the latter part of the lecture. More than that ought probably not to be expected from such a lecture, however, and even so, who would be bold enough to deny that the giving of such lectures and their publication is worth while? If they serve to arouse interest and give their audience even a vague notion of the beauty and significance of scientific progress they have accomplished a noble purpose.

J. W. YOUNG

Principes et Premiers Développements de Géométrie Générale Synthétique Moderne. By Emile Bally. Paris, Gauthier-Villars, 1922. VIII + 218 pp.

The preface to this rather ambitious work contains some amusing statements which may be considered as characteristic of the treatise under review. The author says: "This book is dedicated to the friends of geometry. For many among these, simple amateurs as they are, the ingenious geometric deductions are agreeable relishes which they taste without any afterthought and without bothering themselves with the elements of which they may be composed. Others, more distrustful, the *critics,* wish to know what one has put into the relishes; some of these, the *purists,* cannot tolerate certain ingredients, and declare execrable every composition which contains a trace of those ingredients. Although endeavoring to please the first category, (and this book is above all the

work of an amateur addressed to amateurs) we have tried to take account of the reasonable exigencies of the critics and purists without pretending to give them throughout full satisfaction; a purist is never satisfied."

The author thinks that his treatise can be understood by every intelligent person even without any mathematical knowledge. He does not give any references and seems to excuse this lack by the fact that the only books at his command are the works of Darboux (*Classe Remarquable de Courbes et de Surfaces*), of Dumont (*Surfaces Cubiques*), of Duporcq (*Géométrie Moderne*), and the ENCYCLOPÉDIE DES SCIENCES MATHÉMATIQUES. He also suspects that here and there a proposition supposed to be new may have been discovered or demonstrated before, but leaves the decision as to priority to those who are better versed in the subject.

In a preliminary chapter, M. Bally treats of ordinal arithmetic, then follows Chapter I on "general notions on geometry", in which he includes the fundamental ideas of higher spaces by introducing some new terms, like *polinarity* for the number of elements necessary to determine geometric forms.

On account of the high cost of printing, Chapters II–XI have not been published in the present volume. It concludes with three chapters XII, XIII, XIV on the hexangle, Pascals hexagramme, and related configurations; an appendix on Chapter XIII, and finally with additions and corrections to the previous work of the author on synthetic geometry on unicursal curves of the third class and the fourth order.

The author shows considerable mathematical ability, but he is seriously hampered by his unfamiliarity with the current literature and the present tendencies of geometric research.

ARNOLD EMCH

Darstellende Geometrie, vol. II. By Theodor Schmid. Second edition. Berlin and Leipzig, Walter de Gruyter (Sammlung Schubert, LXVI). 1923. 340 pp.

Although the work of Professor Schmid was to occupy three volumes, three editions of the first and now a second edition of the second volume have appeared before the third volume could be completed. The first edition of volume 2 was published in 1921, and reviewed in this BULLETIN (vol. 28 (1922), pp. 68, 69). The present one differs only slightly from it, and the remarks there made still apply. Three articles have been lengthened; one discusses a nodal normal section of a tubular surface, another amplifies the anaxomatic representation of the helix, and the third enlarges on the projections used in geographic maps. A section has been added on graphical tables and nomography, together with a good bibliography on this subject.

VIRGIL SNYDER

Die mechanischen Beweise für die Bewegung der Erde. By R. Grammel. Berlin, Julius Springer, 1922. IV + 71 pp.

The author states in the preface that his reason for writing the book is to make the material accessible to students who have only an elementary knowledge of mathematics and mechanics. His plan is to describe the experiment, state the law of the motion mathematically, and put the derivation of the formulas in small print. The experiments are grouped into (a) those depending on the principle of the motion of the centroid, (b) those depending on the principle of angular momentum.

Such subjects as Foucault's pendulum experiment, the gyroscopic compass, and the effect of the earth's rotation on a projectile and also on a freely falling particle are familiar to students who have had a course in mechanics. Nevertheless a systematic presentation which gives numerous references and also some experiments not so generally known, makes interesting reading. The material is well arranged and the figures are good.

In connection with Fig. 16, which gives the trajectories of two projectiles fired in vacuo with equal velocity and elevation, one toward the east and the other toward the west, one would naturally wonder if numerical data were used in constructing the diagram, or if it is simply a figure drawn with the idea of bringing out certain general properties.

PETER FIELD

Calculus and Probability for Actuarial Students. By Alfred Henry. Published by the authority of the Institute of Actuaries of Great Britain. London, C. and E. Layton, 1922. 152 pp.

The scope of this book is outlined in its introduction.

"Actuarial science is essentially practical in that, whilst it is based on the processes of pure mathematics, the object of the worker must be to produce a numerical result."

"For this reason it is necessary for considerable prominence to be given, in the curriculum of the actuarial student, to the subject of Finite Differences, and it thus becomes convenient, in the study of those subjects not included under the heading of Algebra, to deal with this part of the syllabus first and, notwithstanding certain theoretical objections, to treat the fundamental propositions of the Differential and the Integral Calculus as being, substantially, special cases of similar propositions in Finite Differences. The subjects enumerated cover so wide a field that it has been necessary to exercise considerable compression."

A chapter on the theory of probability is also added. This book undoubtedly will be the standard introduction to mathematics necessary for a preparation to actuarial science although it may be found rather difficult reading to many students, as graduated exercises and a fuller

discussion of difficulties and of fundamental principles, especially those involving the interrelations between formulas, would clarify the subject. This book coordinates much material that has never been conveniently arranged and should be valuable to students in any field of applied mathematics where interpolation is used or where it is necessary to obtain the approximate summation of the actual data where only typical individual values are known.

J. S. ELSTON

Handbuch der angewandten Mathematik. Herausgegeben von H. E. Timerding. Vol. I. *Praktische Analysis,* von H. von Sanden. Zweite verbesserte Auflage. Leipzig and Berlin, B. G. Teubner, 1923. 195 pp.

The first edition of this book appeared in 1914, and was reviewed in this BULLETIN (vol. 15 (1914–15), pp. 256–259). In the present edition numerous small changes have been made throughout; proofs have been simplified and algebraic work arranged in a manner more easy to follow. A few new sections have been added, in particular: graphical methods now include a discussion of nomography and a larger number of examples, differentiation and integration are derived in connection with formulas for interpolation, expressions for the limit of error in analytic functions are obtained, and a much fuller treatment of numerical integration is added. As in the former edition, both press-work and proof-reading have been well done. Unfortunately the quality of paper used is such that the page has a less pleasing appearance than in the first edition.

VIRGIL SNYDER

Life Contingencies. By E. F. Spurgeon. Published by the authority of the Institute of Actuaries. London, C. and E. Layton, 1922. XXVI + 477 pp.

This book replaces the old Institute of Actuaries Text Book, Part II, which has been the fountain head of knowledge on this subject for thirty-five years in America and England and to some extent in Europe. Of course the new book is more comprehensive, as some additional material is added, the most important being that dealing with functions depending on select lives. The main difference in treatment is that a knowledge of the mathematics covered in the previously reviewed book is assumed throughout, whereas the old book war so arranged that a large part of it could be read without such knowledge. This work will be indispensible to actuaries. As to others, it will be most valuable to anyone interested in statistical subjects connected with the mortality rate.

J. S. ELSTON

NOTES

The third number of volume 24 of the TRANSACTIONS OF THIS SOCIETY (October, 1922) contains the following papers: *On the location of the roots of certain types of polynomials,* by J. L. Walsh; *A note on the preceding paper,* by D. R. Curtiss; *Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials,* by H. S. Everett; *General vector calculus,* by J. B. Shaw.

The Nobel prize in physics for 1923 has been awarded to Professor R. A. Millikan, of the California Institute of Technology.

The Royal Society of London has conferred its Copley medal on Professor Horace Lamb, for his researches in mathematical physics, and its Hughes medal on Professor R. A. Millikan, for his determination of the electronic charge and other physical constants.

The University of Paris has conferred honorary doctorates on Professor S. Arrhenius, of Stockholm, and Sir J. J. Thomson, of Cambridge.

Professor G. D. Birkhoff, of Harvard University, delivered a series of six lectures at the Lowell Institute, Boston, in December, 1923, on *The origin, nature and influence of relativity.*

Professor J. A. Miller, of Swarthmore College, lectured October 25 on *The solar eclipse of* 1923 before the Franklin Institute.

Professor C. W. L. Charlier, of the University of Lund, will lecture at the University of California during the summer of 1924, on *The motion of the stars,* and *The distribution of the stars.*

Dr. J. Trousset, maître de conférences in mathematics at the University of Bordeaux, has been promoted to a professorship.

Mr. R. R. S. Cox has been appointed assistant lecturer and tutor in mathematics at the University of Sheffield.

Dr. B. de Kerékjártó has been appointed lecturer in mathematics at Princeton University.

Mr. H. S. Vandiver, of Cornell University, has been granted leave of absence for research during the second half of the academic year, through a grant from the Heckscher Foundation.

At the University of Vermont, Mr. H. G. Millington has been promoted to an assistant professorship of mathematics in the College of Engineering.

At Swarthmore College, Mr. L. J. Comrie has been promoted from the position of research assistant in the Observatory to an assistant professorship of mathematics and astronomy.

At Lafayette College, Associate Professor W. M. Smith has been granted a six months' leave of absence for study at Oxford and Cambridge.

Mr. F. W. Winters, of Harvard University, has been appointed assistant professor of mathematics at Miami University.

Mr. R. H. MacCullough has been appointed head of the department of mathematics at Defiance College (Ohio).

Dr. H. P. Pettit, of the University of Illinois, has been appointed professor of mathematics and head of the department at Illinois Wesleyan University.

Dr. Marie Whelan, of Johns Hopkins University, has been appointed professor of mathematics at Olivet College (Michigan).

At the Michigan State Normal College, Ypsilanti, Dr. Theodore Lindquist, formerly head of the department of mathematics, State Teachers' College, Emporia, Kansas, has been appointed professor of mathematics, and Associate Professor J. F. Barnhill has been promoted to a full professorship.

At Beloit College, Mr. R. F. Huffer, formerly instructor at Michigan Agricultural College, has been appointed assistant professor.

At Kentucky Wesleyan College, Mr. W. B. Hughes has been appointed to succeed Professor R. G. Demaree as head of the department of mathematics.

At Lander College, Greenwood, South Carolina, Professor W. W. Weber, of Southern College, Lakeland, Florida, has been appointed head of the department of mathematics, and Miss Laura D. Sargent has been appointed associate professor.

Professor C. S. Cox, of Birmingham Southern College, has been appointed head of the department of mathematics and physics at Southern College, Lakeland, Florida.

At Washington University, Assistant Professor Theodore Doll has resigned; Dr. Jessica M. Young has been promoted to an assistant professorship of mathematics and astronomy; and Mr. Edmond Siroky has been appointed assistant professor of applied mathematics.

Associate Professor W. T. Stratton, of Kansas State Agricultural College, has been promoted to a full professorship of mathematics.

At the University of Oklahoma, Professor S. W. Reaves, head of the department of mathematics, has been appointed acting dean of the College of Arts and Sciences. Associate Professor J. O. Hassler has been promoted to a full professorship, and Assistant Professor N. A. Court has been promoted to an associate professorship.

At Oklahoma Agricultural and Mechanical College, Mr. G. B. Grumbine has been appointed professor of mathematics, and Mr. W. C. Payne assistant professor.

At the Agricultural and Mechanical College of Texas, Assistant Professor W. I. Hughes has resigned, and Mr. F. W. Sparks has been appointed assistant professor.

The following have been appointed instructors in mathematics:
Beloit College, Mr. William Oliver and Miss Irene Eldridge;
Case School of Applied Science, Mr. V. L. Benander;
Cornell University, Dr. E. L. Post;
Illinois College, Mr. G. W. Schneider;
Lafayette College, Mr. E. H. Wells;
University of North Dakota, Messrs. S. F. Bibb and J. F. Gates;
University of Tennessee, Mr. Augustus Link;
Agricultural and Mechanical College of Texas, Mr. W. D. Baten and Dr. W. P. Udinski;
Tulane University, Messrs. C. G. Latimer and T. F. Cope;
Union College, Mr. H. N. Rowe;
University of Vermont, Mr. C. H. Nicholson;
Virginia Polytechnic Institute, Mr. T. A. Hatcher;
Washington State College, Mr. O. M. Akey;
Washington University, Mr. H. A. Hoover;
University of West Virginia, Mr. Claire Haskins.

The deaths are announced of Professors P. J. H. Baudet and J. Cardinaal, of the Technical School at Delft.

Professor G. Huber, of the University of Bern, died January 24, 1923, at the age of sixty-five years.

Professor Louis Bertrand, formerly of the College of Geneva, died March 13, 1923, in his eighty-third year.

Charles de Freycinet, member of the Paris Academy of Sciences and of the Académie Française, died May 15, 1923, at the age of ninety-five years. He was the author of a number of mathematical books, and had frequently held high political office.

Professor E. Jablonski, of the Lycée Saint-Louis, Paris, died at Royan in April, 1923.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BOGGIO (T.). Calcolo differenziale con applicazioni geometriche. Volume 1: Funzioni di una variabile. Torino, S. Lattes, 1921. 19 + 611 pp.

BOHR (H.). See ENCYKLOPÄDIE.

BOREL (E.). See JULIA (G.).

BURALI-FORTI (C.) e MARCOLONGO (R.). Corso di matematica per il 2° biennio degli istituti tecnici. Napoli, F. Perella, 1921. 7 + 213 + 8 + 191 pp.

CRAMER (H.). See ENCYKLOPÄDIE.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. Band II 3, Heft 6: N. E. Nörlund, Neuere Untersuchungen über Differenzengleichungen. H. Bohr und H. Cramer, Die neuere Entwicklung der analytischen Zahlentheorie. Leipzig, Teubner, 1922.

FIELD (P.). Projective geometry with applications to engineering. New York, Van Nostrand, 1923. 8 + 98 pp. $2.00

FLAMANT (P.) See JULIA (G.).

HAAG (J.). Exercices du cours de mathématiques spéciales. Tome IV: Géométrie descriptive et trigonométrie. Paris, Gauthier-Villars, 1924. 153 pp.

HAGGE (K.). Die Kunst mathematischer Schönheit und Einfachheit, "Geometrographie". Kiel, 1922. 36 pp.

HAMMER (E.). Lehr- und Handbuch der ebenen und sphärischen Trigonometrie. 5te, erweiterte Auflage. Stuttgart, 1923.

JULIA (G.). Leçons sur les fonctions uniformes à point singulier isolé. Rédigées par P. Flamant. (Collection de Monographies sur la Théorie des Fonctions publiée sous la Direction de M. Emile Borel.) Paris, Gauthier-Villars, 1924. 8 + 151 pp.

KERST (B.). Ebene Geometrie. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1923.

LIETZMANN (W.). Methodik des mathematischen Unterrichts. 2ter Teil. 2te, durchgesehene und vermehrte Auflage. Leipzig, Quelle und Meyer, 1923.

MARCOLONGO (R.). See BURALI-FORTI (C.).

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D'OCAGNE (M.). Notions sommaires de géométrie projective à l'usage des candidats à l'Ecole Polytechnique. Paris, Gauthier-Villars, 1924. 25 pp.

SEVERI (F.). Geometria proiettiva. Padova, La Litotipo, Editrice Universitaria, 1921. 4 + 363 pp.

SHEPPARD (W. F.). From determinant to tensor. Oxford, Clarendon Press, and New York, Oxford University Press, 1923. 127 pp. $2.85

PART II. APPLIED MATHEMATICS

BARR (G.). See GRAETZ (L.).

BURALI-FORTI (C.). Geometria descrittiva. Volume 1: Assonometria. Volume 2: Projezione quotata, projezione Monge, prospettiva. Torino, Lattes, 1921–1922. 15 + 170 + 11 + 144 pp.

CAMPBELL (N. R.). Modern electrical theory. Supplementary chapters. Chapter 17: The structure of the atom. Cambridge, University Press, 1923. 10 + 161 pp.

FRISCHAUF (J.). Grundriss der theoretischen Astronomie und der Geschichte der Planetentheorie. 3te vermehrte Auflage. Leipzig, Engelmann, 1922.

GERLACH (W.). Materie, Elektrizität, Energie; die Entwicklung der Atomistik in den letzten zehn Jahren. Band 7. Dresden und Leipzig, J. Steinkopff, 1923. 195 pp.

GRAETZ (L.). Recent developments in atomic theory. Translated by G. Barr. London, Methuen, 1923. 11 + 174 pp.

HAAS (A.). The new physics. Lectures for laymen and others. New York, Dutton, 1923. 12 + 165 pp.

HATSCHEK (E.). See WILLOWS (R. S.).

KOPFF (A.). Grundzüge der Einsteinschen Relativitätstheorie. 2te Auflage. Leipzig, Hirzel, 1923. 8 + 204 pp.

LACMANN (O.). Die Herstellung gezeichneter Rechentafeln. Ein Lehrbuch der Nomographie. Berlin, Springer, 1923. 8 + 100 pp.

MÜLLER (C. H.) und PRANGE (G.). Allgemeine Mechanik. Hannover, Helwingsche Verlagsbuchhandlung, 1923. 10 + 551 pp.

NUNN (T. P.). Relativity and gravitation. A elementary treatise upon Einstein's theory. London, University of London Press, 1923. 162 pp.

PRANGE (G.). See MÜLLER (C. H.).

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VIAL (F.). Les arguments de M. Einstein. Paris, Gauthier-Villars, 1923. 16 pp.

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